Nonlinear elliptic and parabolic equations related to reaction, diffusion and growth problems

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A mis padres.

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Contents

Introducción

Este trabajo está dedicado a analizar, desarrollar y entender en profundidad un conjunto de problemas, herramientas, o sencillamente preguntas, enmarcadas en la amplia área de las *Ecuaciones en Derivadas Parciales.* En concreto, nos centraremos en una serie de problemas elípticos y parabólicos, todos ellos regidos por un operador surgido del operador elíptico clásico, el Laplaciano.

En líneas generales, trabajaremos con un operador de la forma $(-\Delta)^m$, donde consideraremos potencias fraccionarias, $m \in (0,1)$ (en tal caso denotaremos la potencia con s), pero también $m = 2$. Como veremos, el comportamiento del operador en estos dos casos es completamente distinto, no sólo entre ellos, sino también con respecto al Laplaciano clásico. De hecho, con esta afirmación no nos referimos a dificultades o diferencias de carácter técnico, sino a diferencias en la propia naturaleza del operador.

Sólo por mencionar las dos propiedades más impactantes, en el caso fraccionario el operador tiene un comportamiento no local, mientras que los operadores Δ o Δ^2 son operadores diferenciales y por tanto locales. Por otra parte, cuando tratamos con problemas biarmónicos, surge una limitación muy relevante: no tenemos (en general) principio del máximo. Como puede imaginarse en una primera reflexión, esto imposibilita usar muchas de las técnicas habituales en las EDPs elípticas y parabólicas de segundo orden: la desigualdad de Harnack, argumentos de comparación, métodos tipo Stampacchia...

Sin embargo, a pesar de estas diferencias estructurales, existe un punto en común a lo largo de todo el trabajo: el *Cálculo de Variaciones.* Con este concepto queremos decir lo siguiente (entendiendo esta explicación en el sentido menos riguroso): si tenemos un problema en un dominio acotado Ω,

$$\begin{cases} Lu = f \text{ en } \Omega, \\ \text{condiciones de frontera cero,} \end{cases}$$

donde L es un operador actuando sobre un espacio de Hilbert H, y f pertenece a su espacio dual, la propuesta del Cálculo Variacional será construir un funcional $\mathcal{J} : H \to \mathbb{R}$ de tal manera que encontrar los puntos críticos de $\mathcal{J}$ sea equivalente a resolver el problema.

En otras palabras, trabajar con un funcional cuya ecuación de *Euler-Lagrange* sea la que queremos resolver. Para buscar dichos puntos críticos haremos uso de resultados bien conocidos como el Lema del Paso de la Montaña o diferentes teoremas de minimización. Estas técnicas se basan en conceptos muy abstractos, lo que permite aplicarlas en escenarios muy diferentes.

En cualquier caso, otros métodos como monotonía, bifurcación, o estimaciones a priori, se usarán también en este trabajo.

El Laplaciano fraccionario. Marco no local.

A lo largo de la Parte I y la Parte II de este trabajo, analizaremos en detalle una serie de problemas, elípticos y parabólicos, con una propiedad fundamental en común: todos pueden encuadrarse en el *mundo no local*.

Tras las contribuciones originales de A. P. Calderón y A. Zygmund sobre integrales singulares, un gran número de investigadores contribuyeron al estudio del comportamiento funcional de los operadores pseudodiferenciales, culminando con la teoría general desarrollada por Nirenberg, Kohn, Treves, Hörmander, Fefferman, Stein y Beals, extendida más tarde por Bony, Meyer y Sjöstrand entre otros. Nos referimos, por ejemplo, a los libros de L. Hörmander [122] y M. Taylor [173] para un análisis completo de esta teoría.

Uno de los ejemplos más elementales de operador pseudodiferencial es el Laplaciano fraccionario, cuyo comportamiento en diferentes problemas será el eje central de esta parte de la tesis. Además de en la teoría analítica clásica mencionada anteriormente, este operador tiene importancia en Probabilidad, ya que aparece como un caso particular de procesos de Lévy (véase por ejemplo [38, 43, 124]).

Recientemente, el Laplaciano fraccionario ha cobrado relevancia al surgir en diversos modelos de la Física y otras áreas de aplicación. Por ejemplo, está presente en finanzas [72], problemas de elasticidad [159], propagación de llamas [56], dislocación de cristales [174], o en el problema de membrana delgada [54].

Empezaremos por tanto analizando dicho operador, que puede ser definido de varias formas. En particular, si denotamos por $\mathcal{F}$ la transformada de Fourier y tomamos $u \in \mathcal{S}$, la clase de Schwartz, entonces se tiene

$$\partial_j u = \mathcal{F}^{-1}(i\xi_j \mathcal{F}(u)),$$

y por tanto

$$-\Delta u = \mathcal{F}^{-1}(|\xi|^2 \mathcal{F}(u)).$$

Es decir, *derivar es como multiplicar* en el espacio de Fourier. Por tanto, si queremos calcular una derivada fraccionaria, parece natural hacerlo a través de la transformada de Fourier.

En particular, si $s \in (0,1)$, definiremos el s-Laplaciano fraccionario como

$$(-\Delta)^s u := \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(u)), \tag{0.0.1}$$

y diremos que $|\xi|^{2s}$ es el multiplicador o símbolo del operador. Pero aún así, ¿qué queremos decir con no local?

De forma intuitiva (y poco rigurosa), decimos que un operador es no local cuando, para calcular su valor en un punto, debemos tener en cuenta no sólo dicho punto, sino también más puntos del espacio, incluso a larga distancia. Este hecho queda perfectamente ilustrado en el caso del Laplaciano fraccionario si consideramos la representación de (0.0.1) en términos de la convolución con el núcleo obtenido por la tranformada de Fourier inversa del símbolo. De forma más concreta, podemos ver $(-\Delta)^s$ como

$$
\begin{aligned}
(-\Delta)^s u(x) &= a_{N,s}\, \text{v.p.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy \\
&= a_{N,s} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(0)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy,
\end{aligned}
\tag{0.0.2}
$$

donde queda claro que para calcular el valor de $u(x)$, cada punto $y \in \mathbb{R}^N$ afecta (véase [130, 162] para un profundo análisis de la teoría de integrales singulares). Una prueba de la equivalencia entre esta definición y (0.0.1) puede encontrarse en [130, 162, 166, 177]. La constante de normalización en esta definición es exactamente

$$
a_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1-\cos\xi_1}{|\xi|^{N+2s}}\, d\xi \right)^{-1},
$$

y se elige de forma que se satisface (0.0.1) (véase por ejemplo [82, 160, 166, 177]).

A lo largo del trabajo, veremos cómo la no localidad del operador genera numerosas dificultades que no existen en el caso local. Para lidiar con este comportamiento, L. Caffarelli y L. Silvestre probaron en [57] que de hecho el Laplaciano fraccionario en $\mathbb{R}^N$ puede verse como el valor de frontera de un problema de Neumann local en $\mathbb{R}^{N+1}_+ := \mathbb{R}\times(0,+\infty)$. Nosotros no usaremos aquí esta técnica, así que no profundizaremos en la caracterización rigurosa del operador mediante este método.

Por otra parte, en cuanto a regularidad se refiere, se puede comprobar que, para cada $\phi \in \mathcal{S}$, se cumple

$$
|(-\Delta)^s \phi| \leqslant \frac{C}{1+|x|^{N+2s}},
$$

y entonces, si definimos el espacio

$$
\mathcal{L}^s(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{R} \text{ medible} : \int_{\mathbb{R}^N} \frac{|u(x)|}{|x-y|^{N+2s}}\, dx < +\infty\},
$$

podemos entender el producto de dualidad $\langle (-\Delta)^s u, \phi\rangle$ en sentido distribucional como

$$
\langle (-\Delta)^s u, \phi\rangle := \int_{\mathbb{R}^N} u (-\Delta)^s \phi\, dx,
$$

donde $u \in \mathcal{L}^s(\mathbb{R}^N)$ y $\phi \in \mathcal{S}$. Más aún, a veces esperaremos que nuestras funciones satisfagan los problemas en un sentido más fuerte que el distribucional, concretamente en el marco de *energía finita*, compatible con la transformada de Fourier, dado por los espacios de Sobolev fraccionarios. En concreto, definiremos

$$
\begin{aligned}
H^s(\mathbb{R}^N) &:= \{u \in L^2(\mathbb{R}^N) :\ |\xi|^s \mathcal{F}(u) \in L^2(\mathbb{R}^N)\} \\
&:= \{u \in L^2(\mathbb{R}^N) :\ (-\Delta)^{s/2} u \in L^2(\mathbb{R}^N)\},
\end{aligned}
$$

equipado con la norma

$$
\|u\|_{H^s(\mathbb{R}^N)} := \|u\|_{L^2(\mathbb{R}^N)} + \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}.
$$

Además, si definimos la seminorma de Gagliardo como

$$[u]_{H^s(\mathbb{R}^N)} := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \right)^{1/2},$$

por [82, Proposición 3.6] se satisface

$$\frac{a_{N,s}}{2}[u]^2_{H^s(\mathbb{R}^N)} = \|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^N)}. \tag{0.0.3}$$

De hecho, esta seminorma determinará la formulación de energía de nuestros problemas. En particular, denotaremos

$$\langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} := \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\, dx\, dy.$$

Por otro lado, en la Parte I trataremos con problemas de Dirichlet, es decir, problemas de la forma

$$\begin{cases} (-\Delta)^s u = f & \text{en } \Omega, \\ u = 0 & \text{en } \mathbb{R}^N \setminus \Omega, \end{cases}$$

donde Ω es un dominio suave y acotado de $\mathbb{R}^N$.

Observación 0.0.1. *En este trabajo no nos preocuparemos de la regularidad óptima de Ω, suponiendo tanta regularidad como sea necesaria para justificar los cálculos.*

Nótese que para tener un problema bien definido no es suficiente con prescribir con la condición de frontera en $\partial\Omega$. Esto no es más que otra consecuencia de la naturaleza no local del operador, ya que para calcular el valor de $(-\Delta)^s u$ en cualquier punto de Ω necesitamos saber el valor de u en todo $\mathbb{R}^N$. Entonces, el espacio de Sobolev asociado a estos problemas de Dirichlet será

$$H_0^s(\Omega) := \{u \in H^s(\mathbb{R}^N) \text{ con } u = 0 \text{ c.t.p. de } \mathbb{R}^N \setminus \Omega\},$$

con la norma

$$\|u\|_{H_0^s(\Omega)} := \left(\iint_{Q} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \right)^{1/2},$$

donde

$$Q := \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega).$$

El par $(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)})$ genera un espacio de Hilbert, y además,

$$(-\Delta)^s : H_0^s(\Omega) \to H^{-s}(\Omega)$$

es un operador continuo, donde $H^{-s}(\Omega)$ denota el espacio dual de $H_0^s(\Omega)$. Por tanto, si $u \in H_0^s(\Omega)$ entonces $[u]_{H^s(\mathbb{R}^N)}$ y $\|u\|_{H_0^s(\Omega)}$ coinciden, y de (0.0.3) se deduce

$$\frac{a_{N,s}}{2}\|u\|^2_{H_0^s(\Omega)} = \|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^N)}, \tag{0.0.4}$$

y para cada $\varphi \in H_0^s(\Omega)$,

$$\langle u, \varphi \rangle_{H_0^s(\Omega)} := \iint_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\, dx\, dy.$$

Finalmente, enunciamos la inclusión de Sobolev y el teorema de Rellich-Kondrachov en este marco no local (véase [82]).

Teorema 0.0.1. *(Inclusión de Sobolev fraccionaria).*
Sea $s \in (0,1)$ y $N > 2s$. Entonces, existe una constante $S = S(N,s)$ tal que, para toda función medible y de soporte compacto $\phi : \mathbb{R}^N \to \mathbb{R}$, se tiene

$$\|\phi\|^2_{L^{2_s^*}(\mathbb{R}^N)} \leqslant S \iint_{\mathbb{R}^{2N}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}\, dx\, dy,$$

siendo

$$2_s^* := \frac{2N}{N-2s},$$

el llamado exponente crítico de Sobolev.

Teorema 0.0.2. *(Teorema de Rellich-Kondrachov fraccionario).*
Sea $s \in (0,1)$ y $N > 2s$. Entonces, $H_0^s(\Omega)$ está incluido de forma compacta en $L^p(\Omega)$ para toda $p \in [1, 2_s^*)$.

El potencial de Hardy.

Un objeto que jugará un papel principal en la Parte I de esta tesis es el llamado potencial de Hardy,

$$V(x) := \frac{1}{|x|^2},$$

que surge como una cuestión puramente analítica en dimensión uno (véase [118, 119]). Sin embargo, por razones de integrabilidad, aquí consideraremos el caso $N \geqslant 3$.

De hecho, en este rango de dimensiones el potencial de Hardy aparece por primera vez en el artículo de J. Leray sobre las ecuaciones de Navier-Stokes, [135] (por ello este potencial a menudo recibe el nombre de potencial de Hardy-Leray). Desde el punto de vista de las aplicaciones, surge por ejemplo como un caso límite en Mecánica Cuántica ([60, 144]), en algunos problemas elípticos con términos de reacción supercríticos que son modelos en Teoría de Combustión (véase por ejemplo [62, 108]) o en Astrofísica.

En este contexto, es bien conocida la desigualdad de Hardy clásica,

$$\Lambda_N \int_{\mathbb{R}^N} \frac{\phi^2}{|x|^2} dx \leqslant \int_{\mathbb{R}^N} |\nabla \phi|^2 dx, \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{0.0.5}$$

donde

$$\Lambda_N := \left(\frac{N-2}{2}\right)^2 \tag{0.0.6}$$

es la constante óptima, que no se alcanza.

Nótese que $|x|^{-2} \in L^p_{loc}(\mathbb{R}^N)$ para todo $p < \frac{N}{2}$, y pertenece al espacio de Marcinkiewicz $\mathcal{M}^{\frac{N}{2},\infty}(\mathbb{R}^N)$, es decir, es un caso límite en la teoría de autovalores para el Laplaciano. Esta es la raíz analítica del peculiar comportamiento del potencial de Hardy en su interacción con los operadores diferenciales.

Dicho potencial juega un papel importante para nosotros ya que es posible extender la desigualdad (0.0.5) al caso fraccionario. De hecho, una versión en términos de un multiplicador de Fourier puede escribirse como

Teorema 0.0.3. *(Desigualdad de Hardy fraccionaria).*
Sea $s \in (0,1)$. Para todo $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ se satisface la siguiente desigualdad,

$$\Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx \leqslant \int_{\mathbb{R}^N} |\xi|^{2s} \mathcal{F}^2(u) d\xi,$$

donde

$$\Lambda_{N,s} := 2^{2s} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}.$$

La constante $\Lambda_{N,s}$ es óptima y no se alcanza.

La prueba de este resultado, así como la motivación del potencial de Hardy para el Laplaciano fraccionario, puede encontrarse en [120] (véase también [46, 106, 164, 180]), mientras que la optimalidad y la no alcanzabilidad pueden verse, por ejemplo, en [106, Proposición 4.1]. En particular, por (0.0.4),

$$\Lambda_{N,s} \int_{\Omega} \frac{u^2}{|x|^{2s}}\, dx \leqslant \frac{a_{N,s}}{2} \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy, \qquad u \in H_0^s(\Omega).$$

Observación 0.0.2.

(i) Se puede comprobar que, cuando s tiende a 1,

$$\Lambda_{N,s} \to \Lambda_N,$$

donde Λ_N es la constante de Hardy clásica, definida en (0.0.6).

(ii) Además, reescalando se puede probar que la constante óptima es la misma para cada dominio que contenga el polo del potencial.

Por otra parte, como se puede ver en [102], esta desigualdad puede mejorarse en el siguiente sentido,

$$\frac{a_{N,s}}{2} \|u\|^2_{H_0^s(\Omega)} - \Lambda_{N,s} \int_{\Omega} \frac{u^2}{|x|^{2s}}\, dx \geqslant C(\Omega, q, N, s) \|u\|^2_{W_0^{\tau,2}(\Omega)},$$

para todo $s/2 < \tau < s$ (véase también [9, 93] para una prueba alternativa usando la transformada de Fourier), donde

$$\|u\|_{W_0^{\tau,2}(\Omega)} := \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\tau}}\, dx\, dy \right)^{1/2}.$$

Consúltese [82] para más información sobre estos espacios de Sobolev generales.

Operadores de orden superior. Marco funcional.

En la Parte III de esta tesis dejamos el contexto no local, que conformó el entorno natural de la Parte I y la Parte II, y trabajamos en un marco local.

Concretamente, en el capítulo final cambiamos las potencias fraccionarias del Laplaciano por potencias de orden dos, es decir, trataremos con problemas cuyo operador principal es el bilaplaciano, que, como cabe esperar, se define como

$$\Delta^2 := -\Delta(-\Delta).$$

Desde el punto de vista histórico, el bilaplaciano aparece en el siglo XIX para modelizar una lámina elástica. De hecho, los operadores de cuarto orden juegan un importante papel en diferentes problemas de elasticidad (véase por ejemplo [52, 128, 141, 145, 175]).

Para empezar con el marco funcional de estos problemas, recordemos primero que el espacio de Sobolev $W^{m,p}(\Omega)$ se define como

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : \ D^k u \in L^p(\Omega),\, 1 \leqslant k \leqslant m\},$$

equipado con la norma

$$\|u\|_{W^{m,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \|D^m u\|^p_{L^p(\Omega)} \right)^{1/p},$$

y el espacio $W_0^{m,p}(\Omega)$ como el cierre de $\mathcal{C}_0^\infty(\Omega)$ con respecto a esta norma.

En particular, en este trabajo nos centraremos en el caso $m = p = 2$. De hecho, la forma bilineal

$$(u, v) := \int_\Omega \Delta u \Delta v \, dx,$$

define un producto escalar en $W_0^{2,2}(\Omega)$, cuya norma asociada es

$$\|u\|_{W_0^{2,2}(\Omega)} := \|\Delta u\|_{L^2(\Omega)}.$$

Recordemos también los resultados estándar de inclusión para estos espacios generales (véase por ejemplo [114]).

Teorema 0.0.4. *Supongamos que Ω es un dominio acotado y Lipschitz de $\mathbb{R}^N$. Entonces,*

$$W^{m,p}(\Omega) \subset L^q(\Omega), \text{ para todo } 1 \leqslant q \leqslant \frac{Np}{N - mp}.$$

Además, si $1 \leqslant q < \frac{Np}{N-mp}$ esta inclusión es compacta. Por convenio, $\frac{Np}{N-mp} = +\infty$ si $N < mp$.

En esta parte del trabajo consideraremos solamente problemas en dominios acotados, y por tanto un punto determinante será la condición de frontera. En particular, aunque no son las únicas, nos centraremos en dos casos diferentes: condiciones Dirichlet y condiciones Navier. De forma más precisa, trabajaremos en un problema de la forma

$$\left\{ \begin{array}{ll} \Delta^2 u = f & \text{en } \Omega, \\ B_j(u) = 0 & \text{en } \partial\Omega, \ j \in \{0, 1\}, \end{array} \right. \tag{0.0.7}$$

donde f es una función que satisface determinadas condiciones de integrabilidad.

En el primer caso, condiciones de frontera de tipo Dirichlet, fijamos

$$B_j(u) = \frac{\partial^j u}{\partial \nu^j}, \qquad \text{es decir,} \qquad u = \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega.$$

Entonces, nuestro objetivo será encontrar una solución $u \in W_0^{2,2}(\Omega)$ (el natural para estas condiciones) del correspondiente problema (0.0.7).

En este punto, merece la pena recalcar una las principales características de los operadores de orden alto: no satisfacen (en general) un principio del máximo (esto se ve fácilmente considerando por ejemplo la ecuación biarmónica $u(x) := |x|^2$, que alcanza un mínimo absoluto en el origen). Esto significa que no podemos asegurar que una fuente positiva en el problema implique positividad de la solución, y más aún, que los métodos basados en comparación (métodos iterativos: subsoluciones y supersoluciones) o en truncamientos no funcionan en este contexto. Este hecho, como el lector puede imaginar, convierte a las ecuaciones no lineales de mayor orden en mucho más difíciles de analizar que las análogas de segundo orden.

Sin embargo, dijimos que no tenemos principio del máximo en general, pero puede satisfacerse en casos específicos, como dominios concretos, o determinadas condiciones de frontera. Este es el caso del segundo tipo que consideraremos, las condiciones Navier. Aquí fijamos $B_j(u) = \Delta^j u$, es decir, u tiene que cumplir

$$u = \Delta u = 0 \ \text{ en } \ \partial\Omega.$$

Nótese que en este caso el espacio donde viven nuestras soluciones no es $W_0^{2,2}(\Omega)$, sino

$$V := \{u \in W^{2,2}(\Omega) : \, u = \Delta u = 0 \text{ en } \partial\Omega\}.$$

De hecho, puede verse que

$$V = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

y que es un espacio de Hilbert equipado con el producto escalar

$$(u, v) := \int_\Omega \Delta u \Delta v \, dx, \qquad u, v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

cuya norma inducida es equivalente a $\| \cdot \|_{W^{2,2}(\Omega)}$ (véase [114, Teorema 2.31] para la demostración). Entonces, para los dos tipos de condiciones de frontera, trabajaremos con la norma $\|\Delta u\|_{L^2(\Omega)}$.

Pero volviendo a la propiedad de preservar la positividad en el caso de condiciones Navier, se puede comprobar fácilmente que resolver el problema

$$\begin{cases} \Delta^2 u = f & \text{en } \Omega, \\ u = \Delta u = 0 & \text{en } \partial\Omega, \end{cases}$$

es equivalente a resolver el sistema

$$\begin{cases} -\Delta u = v & \text{en } \Omega, \\ u = 0 & \text{en } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v = f & \text{en } \Omega, \\ v = 0 & \text{en } \partial\Omega. \end{cases}$$

Es decir, hemos separado nuestro problema Navier de orden cuatro en dos problemas de Dirichlet de orden dos, cuyo operador principal es el Laplaciano, y donde podemos aplicar el principio del máximo y todas las herramientas asociadas. Como veremos, esta diferencia entre las condiciones Dirichlet y Navier determinará las técnicas que usaremos en cada caso.

Además, debido a la no linealidad de los problemas que consideraremos, podremos o no usar formulación variacional en función de las condiciones de frontera.

Organización, resultados principales y conclusiones.

Finalmente, resumimos brevemente la organización de este trabajo y los principales resultados contenidos en cada capítulo. La tesis está formada por cinco capítulos, agrupados en tres partes.

Parte I: Problemas singulares no locales en dominios acotados.

La primera parte engloba el estudio de varios problemas de carácter no local en dominios acotados, con el Laplaciano fraccionario como operador principal, con un punto en común: en el lado derecho de todos ellos aparece un término singular, aunque éste puede ser de muy diferente naturaleza. En esta parte del trabajo, suponemos siempre $s \in (0, 1)$.

En el **Capítulo 1** estudiamos un problema elíptico cóncavo-convexo donde el potencial de Hardy interfiere con el Laplaciano fraccionario. En particular, estudiaremos el problema

$$(P_{\lambda,\mu}) \begin{cases} (-\Delta)^s u - \lambda \dfrac{u}{|x|^{2s}} = u^p + \mu u^q & \text{en } \Omega, \\ \qquad\qquad u > 0 & \text{en } \Omega, \\ \qquad\qquad u = 0 & \text{en } \mathbb{R}^N \setminus \Omega\,, \end{cases} \tag{0.0.8}$$

donde $0 \in \Omega$, $N > 2s$, $\mu > 0$, $0 < q < 1$, $0 < \lambda < \Lambda_{N,s}$ y $p > 1$.

En primer lugar, mediante un argumento de comparación, veremos que todas las soluciones de este problema son singulares en el origen. Este hecho está estrechamente relacionado con el comportamiento de las soluciones radiales del problema homogéneo en $\mathbb{R}^N$, es decir,

$$(-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} \quad \text{en } \mathbb{R}^N \setminus \{0\}, \tag{0.0.9}$$

lo que justifica el profundo análisis de estas soluciones que realizaremos en este capítulo.

A continuación, encontraremos un umbral $p(\lambda, s)$ tal que para $1 < p < p(\lambda, s)$ podemos encontrar al menos una solución positiva del problema, mientras que para $p > p(\lambda, s)$ probaremos no sólo no existencia, sino también *blow-up* completo e instantáneo, donde la obligada singularidad de las posibles soluciones tendrá un rol fundamental. También veremos no existencia cuando $\lambda > \Lambda_{N,s}$.

De hecho, en el caso $1 < p < p(\lambda, s)$, mediante técnicas de monotonía podemos probar la existencia de una solución positiva para cada $0 < \mu \leqslant M < +\infty$, donde M se define como

$$M := \sup\{\mu > 0 : \text{ el problema } (P_{\lambda,\mu}) \text{ tiene al menos una solución}\}.$$

Como veremos, es natural diferenciar el tipo de soluciones dependiendo del rango de p. Si $1 < p \leqslant 2_s^* - 1$ consideraremos soluciones de energía, mientras que para $2_s^* - 1 < p < p(\lambda, s)$ trabajaremos con soluciones en sentido *distribucional* (véase la Definición 1.1.3). Además, en el caso $1 < p \leqslant 2_s^* - 1$, aplicando técnicas variacionales probaremos la existencia de una segunda solución, primero cuando μ es suficientemente pequeño, y finalmente, de forma global, para cada $0 < \mu < M$.

Los resultados de este primer capítulo pueden encontrarse en [34].

El **Capítulo 2** está dedicado a analizar el problema parabólico homólogo al del Capítulo 1. En particular, consideraremos los problemas

$$\text{(0.0.10)} \qquad \begin{cases} u_t + (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + f & \text{en } \Omega \times (0, T), \\ u(x,t) > 0 & \text{en } \Omega \times (0, T), \\ u(x,t) = 0 & \text{en } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\ u(x,0) = u_0(x) & \text{si } x \in \Omega, \end{cases}$$

y

$$\text{(0.0.11)} \qquad \begin{cases} u_t + (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + u^p + f & \text{en } \Omega \times (0, T), \\ u(x,t) > 0 & \text{en } \Omega \times (0, T), \\ u(x,t) = 0 & \text{en } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\ u(x,0) = u_0(x) & \text{si } x \in \Omega, \end{cases}$$

donde $N > 2s$, $p > 1$, y c y λ son constantes positivas, $0 \in \Omega$, y f y u_0 son funciones no negativas que satisfacen ciertas condiciones de sumabilidad.

Para el primer problema, estableceremos condiciones necesarias y suficientes en la integrabilidad de g y u_0 con el fin de probar existencia de solución. Tales condiciones dependerán de λ, a través de la singularidad de las soluciones radiales de (0.0.9). Más concretamente, necesitaremos probar que nuestras soluciones se comportan en el origen exactamente como estas funciones radiales.

De hecho, este punto convierte este capítulo en una extensión no trivial del artículo clásico para la ecuación del calor de P. Baras y J. Goldstein [44], ya que en el caso no local la prueba de la singularidad en el origen es mucho más complicada que en el caso elíptico. La estrategia será transformar nuestro problema en uno nuevo,

$$\begin{cases} |x|^{-2\gamma} v_t + L_\gamma v = |x|^{-\gamma} f(x,t) & \text{en } \Omega \times (0, T), \\ v(x,t) = 0 & \text{en } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\ v(x,0) = v_0(x) := |x|^\gamma u_0(x) & \text{si } x \in \Omega, \end{cases}$$

donde L_γ es un operador con pesos, de la forma

$$L_\gamma(v(x,t)) := a_{N,s}\,\text{v.p.}\int_{\mathbb{R}^N}\frac{v(x,t)-v(y,t)}{|x-y|^{N+2s}}\frac{dx}{|x|^\gamma}\frac{dy}{|y|^\gamma},$$

que aparece de forma natural al escribir la ecuación para $v(x,t) := |x|^{-\gamma}u(x,t)$, siendo γ la potencia de las soluciones radiales de (0.0.9). Entonces, la singularidad de las soluciones de (0.0.10) será una consecuencia de ver que estos operadores con pesos satisfacen una desigualdad de Harnack. De hecho, gran parte del Capítulo 2 está dedicado a probar este resultado.

En cuanto al problema semilineal (0.0.11), probaremos que se mantiene el mismo comportamiento en cuanto a existencia de solución que en el caso elíptico (Capítulo 1). En particular, con argumentos de comparación probaremos la existencia de al menos una solución para cada $1 < p < p(\lambda, s)$ (que de nuevo será de energía o débil en función del valor de p), y no existencia y *blow-up* completo para $p > p(\lambda, s)$. Conviene recalcar que esta barrera $p(\lambda, s)$ es exactamente la misma que en el caso elíptico.

Los resultados contenidos en el Capítulo 2 se pueden encontrar en [2].

Finalmente, en el último capítulo de la Parte I, el **Capítulo 3**, consideramos de nuevo un problema elíptico singular, pero de naturaleza muy diferente. En este caso, la dificultad viene dada por un término no lineal que es singular en la frontera en lugar de en el origen. Más concretamente, estudiaremos la existencia de soluciones del problema

$$(D_{\mu,\alpha,p})\begin{cases}(-\Delta)^s u = \dfrac{f(x)}{u^\alpha} + Mu^p & \text{en } \Omega,\\ u > 0 & \text{en } \Omega,\\ u = 0 & \text{en } \mathbb{R}^N \setminus \Omega,\end{cases}$$

en función del valor de α. En este caso, $N > 2s$, $M \in \{0, 1\}$, $\alpha > 0$, $p > 1$, y f es una función no negativa.

En el caso $M = 0$, probaremos la existencia de una solución de energía si $0 < \alpha \leqslant 1$ y f tiene una integrabilidad adecuada, y de solución débil en el caso $\alpha > 1$ y $f \in L^1(\Omega)$. La idea para encontrar estas soluciones será trabajar con los problemas truncados, pasando al límite al final.

Cuando $M = 1$, por un argumento de monotonía más delicado, encontraremos también una solución para todo $p > 1$, y para todo $\mu \in (0, \Upsilon)$, donde

$$\Upsilon := \sup\{\mu > 0 \text{ tal que el problema } (D_{\mu,\alpha,p}) \text{ tiene al menos una solución}\}.$$

Veremos que de hecho $\Upsilon < +\infty$.

Para terminar este capítulo, en la Sección 3.4 consideraremos el problema

$$\begin{cases}(-\Delta)^s u = \lambda\dfrac{u}{|x|^{2s}} + \dfrac{h(x)}{u^\alpha} & \text{en } \Omega,\\ u > 0 & \text{en } \Omega,\\ u = 0 & \text{en } \mathbb{R}^N \setminus \Omega,\end{cases}$$

con h una función no negativa, y $\alpha > 0$. Obsérvese que estudiar la solubilidad de este problema es en cierto modo analizar el intervalo de potencias complementario (es decir, exponentes negativos) en el problema (0.0.8), donde se consideraron potencias positivas.

En particular, como en $(D_{\mu,\alpha,p})$ para $M = 0$, probaremos existencia de solución de energía para $\alpha \leqslant 1$ y h suficientemente integrable, y de solución débil si $\alpha > 1$ y $h \in L^1(\Omega)$. Además, en el caso $\alpha < 1$, encontraremos una solución débil pidiendo solamente $h \in L^1(\Omega, |x|^{-(1-\alpha)\gamma})$, donde γ es de nuevo el crecimiento en el origen de las soluciones radiales de (0.0.9). Como sugiere este exponente, para probar este resultado haremos uso de la teoría desarrollada en el Capítulo 2 para los espacios con pesos.

Los resultados de las Secciones 3.1 - 3.3 de este capítulo pueden encontrarse en [33]. La Sección 3.4 aparecerá en [3].

Parte II: Resultados de bifurcación para una ecuación crítica no local en $\mathbb{R}^N$.

La segunda parte de este trabajo trata aún con problemas de índole no local, pero en este caso trabajaremos en todo $\mathbb{R}^N$ en lugar de en un dominio acotado. Además, consideraremos un problema elíptico semilinear sin término singular, pero de crecimiento crítico. Más concretamente, a lo largo del **Capítulo 4** estudiaremos el problema

$$(-\Delta)^s u = \varepsilon\, h\, u^q + u^p \quad \text{en } \mathbb{R}^N,$$

donde $s \in (0, 1)$, $N > 4s$, $\varepsilon > 0$ es un parámetro pequeño, $p = 2_s^* - 1$, $0 < q < p$ y h es una función continua que satisface

$$\omega := sop\, h \text{ es compacto} \qquad \text{y} \qquad h_+ \not\equiv 0.$$

En particular, probaremos la existencia de una solución $u_{1,\varepsilon}$ de este problema, considerándolo una perturbación de la ecuación

$$(-\Delta)^s u = u^p \quad \text{en } \mathbb{R}^N, \qquad p = 2_s^* - 1. \tag{0.0.12}$$

De hecho, $u_{1,\varepsilon}$ tenderá a una de las soluciones de (0.0.12) cuando $\varepsilon \to 0$, que son precisamente los minimizantes de la inclusión de Sobolev. Si h cambia de signo, probaremos la existencia de una segunda solución, que convergerá a un minimizante diferente cuando $\varepsilon \to 0$.

Para obtener estos resultados de existencia, realizaremos una reducción de Lyapunov-Schmidt, aprovechándonos de la estructura variacional del problema.

Los resultados de este capítulo pueden encontrarse en [84].

Parte III: Problemas elípticos biarmónicos.

En la última parte, que corresponde al **Capítulo 5**, estudiaremos diferentes problemas locales en dimensión $N = 3$.

En particular, consideraremos un operador de cuarto orden, el bilaplaciano, y varios problemas no lineales, cuyo modelo será

$$
\begin{cases} \Delta^2 u = S_2(D^2 u), & \text{en } \Omega \subset \mathbb{R}^N, \\ B(u) = 0, & \text{en } \partial\Omega. \end{cases} \tag{0.0.13}
$$

Aquí $B(u)$ denota unas condiciones de frontera genéricas, y

$$
S_2(D^2 u)(x) := \sum_{1 \leqslant i < j \leqslant N} \lambda_i(x)\lambda_j(x),
$$

siendo λ_i, con $i = 1, \dots, N$, los autovalores de la matriz Hessiana, es decir, las soluciones a la ecuación

$$
\det\big(\lambda I - D^2 u(x)\big) = 0.
$$

En el caso de condiciones de frontera de tipo Dirichlet podemos encontrar un funcional de energía de forma que nuestro problema es la ecuación de *Euler-Lagrange* asociada. Entonces, mediante técnicas variacionales probaremos la existencia de al menos una solución.

Además, veremos cómo añadir un término de la forma $\mu|u|^{p-1}u$ determina la multiplicidad de soluciones: para $p < 1$ encontraremos dos soluciones si μ es suficientemente pequeño; para $p > 1$ probaremos la existencia de al menos una solución para cada $\mu \geqslant 0$; y para el caso lineal, $p = 1$, probaremos existencia de solución cuando μ es menor que el primer autovalor de Δ^2 en Ω.

Cuando trabajamos con condiciones de frontera de tipo Navier, no es posible un planteamiento variacional, y de hecho no sabemos nada sobre la solubilidad del problema (0.0.13). En cualquier caso, aplicando el teorema de bifurcación de Rabinowitz, veremos que existe una rama no acotada de soluciones del problema

$$
\begin{cases} \Delta^2 u = S_2(D^2 u) + \lambda u, & \text{en } \Omega, \\ \quad u = 0, & \text{en } \partial\Omega, \\ \ \Delta u = 0, & \text{en } \partial\Omega, \end{cases}
$$

bifurcando del primer autovalor.

Finalmente, consideraremos un último problema con condiciones Navier,

$$
\begin{cases} \Delta^2 u = S_2(D^2 u) + \mu f(x) & \text{en } \Omega, \\ \quad u = 0 & \text{en } \partial\Omega, \\ \ \Delta u = 0 & \text{en } \partial\Omega. \end{cases}
$$

En este caso, probaremos la existencia de al menos una solución para $f \in L^1(\Omega)$ y μ suficientemente pequeño, mediante un argumento de punto fijo.

Los resultados de este capítulo pueden encontrarse en [100].

Introduction

This work is devoted to analyze, develop and deeply understand a set of problems, tools, or simply questions framed in the huge field of *Partial Differential Equations*. In particular, we will focus on elliptic and parabolic problems, all of them led by an operator emerged from the standard elliptic operator, the Laplacian.

Roughly speaking, we will work with an operator of the form $(-\Delta)^m$, where we will consider fractional powers, $m \in (0,1)$ (in this case we will denote the power by s), but also $m = 2$. As we will see, the behavior of the operator in these two cases is completely different, not only between them, but also with respect to the classical Laplacian. Actually, with this assertion we are not talking about technical differences or difficulties, we mean differences in the proper nature of the operator.

Just to mention the two most *shocking* points, in the fractional case the operator has nonlocal behavior, while the operators Δ or Δ^2 are pseudodifferential operators and thus nonlocal. On the other hand, when we deal with biharmonic problems, a very relevant limitation arises: we do not have (in general) a maximum principle. As one can imagine just in a first thought, this implies the impossibility of using most of the standard techniques on second order elliptic and parabolic PDE's: Harnack's inequality, comparison arguments, Stampacchia-type methods...

Nevertheless, despite of these structural differences, there will be a common point throughout the work: the *Calculus of Variations*. By this concept we mean the following (understanding this explanation in the least rigourous sense): if we have a problem on a bounded domain Ω,

$$\begin{cases} Lu = f \text{ in } \Omega, \\ \text{zero boundary conditions,} \end{cases}$$

where L is an operator acting on some Hilbert space H, and f belongs to its dual space, the proposal of Calculus of Variations will be to build a functional $\mathcal{J} : H \to \mathbb{R}$ such that finding critical points of $\mathcal{J}$ is equivalent to solving the problem.

In other words, to deal with a functional whose *Euler-Lagrange equation* is the one we want to solve. To search for these critical points we can make use of well known results, like the Mountain Pass Lemma, or different minimization theorems. These techniques work in a very abstract setting, what allows us to apply them even in very different scenarios.

In any case, other approaches, like monotonicity, bifurcation, or a priori estimates, will be also used in this work.

The fractional Laplacian. Nonlocal setting.

Along Part I and Part II of this work, we will analyze in detail a set of problems, both elliptic and parabolic, with a common fundamental property: they can all be framed in the *nonlocal world*.

After the original contributions of A. P. Calderón and A. Zygmund about singular integrals, a large number of researchers contributed to the study of the functional behavior of the pseudodifferential operators, culminating with the general theory developed by Nirenberg, Kohn, Treves, Hörmander, Fefferman, Stein and Beals, and later extended by Bony, Meyer and Sjöstrand among others. We refer, for instance, to the books of L. Hörmander [122] and M. Taylor [173] for a complete analysis of this theory.

One of the most elementary examples of pseudodifferential operator is the fractional Laplacian, whose behavior in different problems will be the core of this part of the thesis. Apart from the analytical classical theory previously mentioned, this operator is important in Probability, since it appears as a particular case of Lévy processes (see for example [38, 43, 124]).

Recently, the fractional Laplacian has become more relevant after arising in some models in Physics and other fields of application. For instance, it is present in finance [72], elasticity problems [159], flames propagation [56], crystal dislocation [174], or the thin obstacle problem [54].

Thus, let us start by analyzing such operator, that can be defined in several ways. In particular, if we denote the Fourier transform by $\mathcal{F}$ and we take $u \in \mathcal{S}$, the Schwartz class, then there holds

$$\partial_j u = \mathcal{F}^{-1}(i\xi_j \mathcal{F}(u)),$$

and thus

$$-\Delta u = \mathcal{F}^{-1}(|\xi|^2 \mathcal{F}(u)).$$

That is, *differentiating means multiplying* in the Fourier space. Therefore, if we want to compute a fractional derivative, it seems natural to do it by means of the Fourier transform.

In particular, if $s \in (0,1)$, we will define the s-fractional Laplacian as

$$(-\Delta)^s u := \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(u)), \tag{0.0.14}$$

and we will say that $|\xi|^{2s}$ is the multiplier or symbol of the operator. But still, what do we mean by nonlocal?

Roughly speaking, we say that an operator is nonlocal when, to compute its value at any point, we have to take into account more points around, even if they are far away. This fact is illustrated in the case of the fractional Laplacian if we consider the representation of (0.0.14) in terms of the convolution with the kernel obtained through the inverse Fourier transform of the symbol. More precisely, we can see $(-\Delta)^s$ as

$$\begin{aligned} (-\Delta)^s u(x) &= a_{N,s}\, \text{p.v.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy \\ &= a_{N,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \end{aligned} \tag{0.0.15}$$

where it is clear that to compute the value at $u(x)$, every point $y \in \mathbb{R}^N$ counts (see [130, 162] for a deep analysis of the theory of singular integrals). A proof of the equivalence between this definition and (0.0.14) can be found in [130, 162, 166, 177]. Moreover, through a standard change of variables, this operator can be seen as a second incremental quotient,

$$(-\Delta)^s u(x) = \frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2s}}\, dy.$$

The normalization constant in these definitions is exactly

$$a_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1 - \cos \xi_1}{|\xi|^{N+2s}}\, d\xi \right)^{-1},$$

and it is chosen so that (0.0.14) holds (see for example [82, 160, 166, 177]).

Throughout this work, we will see how the nonlocality of the operator generates many difficulties which do not exist in the classical case. To face this behavior, L. Caffarelli and L. Silvestre proved in [57] that the fractional Laplacian in $\mathbb{R}^N$ can indeed be seen as the boundary value of a local Neumann problem in $\mathbb{R}^{N+1}_+ := \mathbb{R} \times (0, +\infty)$. We will never use this technique here, so we skip the rigorous characterization of the operator by means of this method.

On the other hand, concerning the regularity, it can be checked that, for every $\phi \in \mathcal{S}$, there holds

$$|(-\Delta)^s \phi| \leqslant \frac{C}{1 + |x|^{N+2s}},$$

and thus, if we define the space

$$\mathcal{L}^s(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} \frac{|u(x)|}{|x-y|^{N+2s}}\, dx < +\infty\},$$

we can understand the duality product $\langle (-\Delta)^s u, \phi \rangle$ in distributional sense as

$$\langle (-\Delta)^s u, \phi \rangle := \int_{\mathbb{R}^N} u (-\Delta)^s \phi \, dx,$$

whenever $u \in \mathcal{L}^s(\mathbb{R}^N)$ and $\phi \in \mathcal{S}$. Furthermore, sometimes we will expect our functions to satisfy the problems in a sense stronger than distributional, specifically in the *finite energy* framework, compatible with the Fourier transform, given by the fractional Sobolev spaces. Actually, we define

$$\begin{aligned} H^s(\mathbb{R}^N) &:= \{u \in L^2(\mathbb{R}^N) : \ |\xi|^s \mathcal{F}(u) \in L^2(\mathbb{R}^N)\} \\ &:= \{u \in L^2(\mathbb{R}^N) : \ (-\Delta)^{s/2} u \in L^2(\mathbb{R}^N)\}, \end{aligned}$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} := \|u\|_{L^2(\mathbb{R}^N)} + \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}.$$

Furthermore, if we define the Gagliardo seminorm

$$[u]_{H^s(\mathbb{R}^N)} := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \right)^{1/2},$$

by [82, Proposition 3.6] it satisfies

$$\frac{a_{N,s}}{2}[u]^2_{H^s(\mathbb{R}^N)} = \|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^N)}. \tag{0.0.16}$$

Indeed, this seminorm will determine the energy formulation of our problems. In particular, we will denote

$$\langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} := \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\, dx\, dy.$$

On the other hand, in Part I we will deal with Dirichlet problems, that is, problems of the form

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$.

Remark 0.0.3. *In this work we will not be concerned with the optimal regularity of Ω, assuming as much regularity as needed to justify the computations.*

Notice that to have a well defined problem it is not enough to prescribe the boundary condition at $\partial\Omega$. This is nothing but another consequence of the nonlocal nature of the operator, since to compute the value of $(-\Delta)^s u$ at any point in Ω we need to know the value of u in the whole $\mathbb{R}^N$. Thus, the Sobolev space associated to these Dirichlet problems will be

$$H^s_0(\Omega) := \{u \in H^s(\mathbb{R}^N) \text{ with } u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

endowed with the norm

$$\|u\|_{H^s_0(\Omega)} := \left(\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \right)^{1/2},$$

where

$$Q := \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega).$$

The pair $(H^s_0(\Omega), \|\cdot\|_{H^s_0(\Omega)})$ yields a Hilbert space, and moreover,

$$(-\Delta)^s : H^s_0(\Omega) \to H^{-s}(\Omega)$$

is a continuous operator, where $H^{-s}(\Omega)$ denotes the dual space of $H^s_0(\Omega)$. Hence, if $u \in H^s_0(\Omega)$ then $[u]_{H^s(\mathbb{R}^N)}$ and $\|u\|_{H^s_0(\Omega)}$ coincide, and from (0.0.16) it yields

$$\frac{a_{N,s}}{2}\|u\|^2_{H^s_0(\Omega)} = \|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^N)}, \tag{0.0.17}$$

and for every $\varphi \in H_0^s(\Omega)$,

$$\langle u, \varphi \rangle_{H_0^s(\Omega)} := \iint_Q \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy.$$

Finally, we state the fractional version of the Sobolev embedding and the Rellich-Kondrachov theorem (see [82]).

Theorem 0.0.4. *(Fractional Sobolev embedding).*
Let $s \in (0,1)$ and $N > 2s$. Then, there exists a constant $S = S(N,s)$ such that, for any measurable and compatcly supported function $\phi : \mathbb{R}^N \to \mathbb{R}$, there holds

$$\|\phi\|^2_{L^{2^*_s}(\mathbb{R}^N)} \leqslant S \iint_{\mathbb{R}^{2N}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dx \, dy,$$

being

$$2^*_s := \frac{2N}{N - 2s},$$

the so called fractional critical exponent.

Theorem 0.0.5. *(Fractional Rellich-Kondrachov theorem).*
Let $s \in (0,1)$ and $N > 2s$. Then, $H_0^s(\Omega)$ is compactly embedded in $L^p(\Omega)$ for every $p \in [1, 2^*_s)$.

The Hardy potential.

An object that will perform a leading role in Part I of this thesis is the so called Hardy potential,

$$V(x) := \frac{1}{|x|^2},$$

which appears as a pure analytical subject in one dimension (see [118, 119]). Nevertheless, because of integrability reasons, we will consider the case $N \geqslant 3$.

Indeed, in this range of dimensions the Hardy potential arises for the first time in the seminal paper by J. Leray about the Navier-Stokes equations, [135] (this is why this potential is often called Hardy-Leray potential). From the point of view of applications, it appears for instance as a borderline case in Quantum Mechanics ([60, 144]), in some elliptic problems with supercritical reaction terms that are models in Combustion Theory (see for example [62, 108]) or in Astrophysics.

In this setting, the classical Hardy inequality,

$$\Lambda_N \int_{\mathbb{R}^N} \frac{\phi^2}{|x|^2} dx \leqslant \int_{\mathbb{R}^N} |\nabla \phi|^2 dx, \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{0.0.18}$$

it is well known, where

$$\Lambda_N := \left(\frac{N-2}{2} \right)^2 \tag{0.0.19}$$

is the optimal constant, which is not attained.

Notice that $|x|^{-2} \in L^p_{loc}(\mathbb{R}^N)$ for all $p < \frac{N}{2}$, and it belongs to the Marcinkiewicz space $\mathcal{M}^{\frac{N}{2},\infty}(\mathbb{R}^N)$. This is the analytical root of the peculiar behavior of the Hardy potential in its interaction with the differential operators.

This potential plays an important role for us since inequality (0.0.18) can be extended to the fractional case. Indeed, a classical extension of the Hardy inequality in terms of the Fourier multiplyer can be written as follows,

Theorem 0.0.6. *(Fractional Hardy inequality).*
Let $s \in (0,1)$. For all $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ the following inequality holds,

$$\Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx \leqslant \int_{\mathbb{R}^N} |\xi|^{2s} \mathcal{F}^2(u) d\xi, \tag{0.0.20}$$

where

$$\Lambda_{N,s} := 2^{2s} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}.$$

The constant $\Lambda_{N,s}$ is optimal and not attained.

The proof of this result, and the motivation to treat with the Hardy potential in the fractional case, can be found in [120] (see also [46, 106, 164, 180]), while the optimality and nonattainability of the constant may be seen, for instance, in [106, Proposition 4.1]. In particular, by (0.0.17),

$$\Lambda_{N,s} \int_\Omega \frac{u^2}{|x|^{2s}}\, dx \leqslant \frac{a_{N,s}}{2} \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy, \qquad u \in H_0^s(\Omega).$$

Remark 0.0.7.

(i) It can be checked that, when s tends to 1,

$$\Lambda_{N,s} \to \Lambda_N,$$

where Λ_N is the classical Hardy constant, defined in (0.0.19)*.*

(ii) Moreover, by scaling it can be proved that the optimal constant is the same for every domain that contains the pole of the potential.

Furthermore, as can be seen in [102], this inequality can be improved in the following sense,

$$\frac{a_{N,s}}{2} \|u\|^2_{H_0^s(\Omega)} - \Lambda_{N,s} \int_\Omega \frac{u^2}{|x|^{2s}}\, dx \geqslant C(\Omega, q, N, s) \|u\|^2_{W_0^{\tau,2}(\Omega)}, \tag{0.0.21}$$

for all $s/2 < \tau < s$ (see also [9, 93] for an alternative proof without using the Fourier transform), where

$$\|u\|_{W_0^{\tau,2}(\Omega)} := \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\tau}}\, dx\, dy \right)^{1/2}.$$

See [82] for more information about these general Sobolev spaces.

Higher order operators. Functional framework.

In Part III of this thesis we forget about the nonlocal framework, that was the natural environment of Part I and Part II, and we work in a local setting.

Indeed, in the final Chapter we change the fractional powers of the Laplacian by second order powers, that is, we will deal with problems whose main operator is the bilaplacian, that, as one may expect, is defined as

$$\Delta^2 := -\Delta(-\Delta).$$

From the historical point of view, the bilaplacian appears in the XIX century to model an elastic plate. Indeed, fourth order operators play an important role in several elasticity problems (see for example [52, 128, 141, 145, 175]).

To begin with the functional framework for these problems, let us recall first that we define the Sobolev space $W^{m,p}(\Omega)$ as

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : \ D^k u \in L^p(\Omega),\, 1 \leqslant k \leqslant m\},$$

endowed with the norm

$$\|u\|_{W^{m,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \|D^m u\|^p_{L^p(\Omega)}\right)^{1/p},$$

and the space $W_0^{m,p}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to this norm.

In particular, in this work we will be concerned with the case $m = p = 2$. In fact, the bilinear form

$$(u, v) := \int_\Omega \Delta u \Delta v \, dx,$$

defines a scalar product in $W_0^{2,2}(\Omega)$, whose associated norm is

$$\|u\|_{W_0^{2,2}(\Omega)} := \|\Delta u\|_{L^2(\Omega)}.$$

Let us also recall the standard embedding results for these general spaces.

Theorem 0.0.8. *Assume that Ω is a Lipschitz bounded domain of $\mathbb{R}^N$. Then,*

$$W^{m,p}(\Omega) \subset L^q(\Omega), \ \textit{for any } 1 \leqslant q \leqslant \frac{Np}{N-mp}.$$

Furthermore, if $1 \leqslant q < \frac{Np}{N-mp}$ this embedding is compact. We make the convention $\frac{Np}{N-mp} = +\infty$ if $N < mp$.

In this part of the work we will only consider problems in bounded domains, and thus a determining point will be the boundary condition. In particular, although they are not the only ones, we will focus on two different cases: Dirichlet conditions and Navier conditions. More precisely, we will work on a problem of the form

$$\left\{\begin{array}{ll} \Delta^2 u = f & \text{in } \Omega, \\ B_j(u) = 0 & \text{on } \partial\Omega, \ j \in \{0,1\}, \end{array}\right. \tag{0.0.22}$$

where f is a function with suitable summability conditions.

In the first case, Dirichlet boundary conditions, we set

$$B_j(u) = \frac{\partial^j u}{\partial \nu^j}, \qquad \text{i.e.,} \qquad u = \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega.$$

Thus, our goal will be to find a solution $u \in W_0^{2,2}(\Omega)$ (the natural space for these conditions) of the corresponding problem (0.0.22).

At this point, it is worth to point out one of the main features of higher order operators: they do not satisfy a maximum principle in general (this can be easily seen for instance with the biharmonic equation $u(x) := |x|^2$, that attains an absolut minimum at the origin). This means that we cannot assure that a positive source in our problem implies positivity in our solution, and furthermore, that methods based on comparison (iterative methods: subsolutions and supersolutions) or truncation do not work in this framework. This fact, as the reader can imagine, makes the nonlinear higher order equations pretty much more difficult to analyze than the analogous second order equations.

However, although we said that we do not have a maximum principle in general, it may hold in special cases, like concrete domains, or specific boundary conditions. This is the case of the second type we will consider, Navier boundary conditions. Here we set $B_j(u) = \Delta^j u$, that is, u has to satisfy

$$u = \Delta u = 0 \ \text{ on } \ \partial\Omega.$$

Notice that in this case the space where our solutions live is not $W_0^{2,2}(\Omega)$, but

$$V := \{u \in W^{2,2}(\Omega) : \, u = \Delta u = 0 \text{ on } \partial\Omega\}.$$

Indeed, it can be seen that

$$V = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

and that it is a Hilbert space endowed with the scalar product

$$(u, v) := \int_\Omega \Delta u \Delta v \, dx, \qquad u, v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

whose induced norm is equivalent to $\| \cdot \|_{W^{2,2}(\Omega)}$ (see [114, Theorem 2.31] for a proof). Thus, for both types of boundary conditions, we will work with the norm $\|\Delta u\|_{L^2(\Omega)}$.

But coming back again to the positivity preserving property in the case of Navier conditions, it can be easily checked that solving the problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.23}$$

is equivalent to solving the system

$$\begin{cases} -\Delta u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

That is, we have split our fourth-order Navier problem into two second-order Dirichlet problems, whose main operator is the Laplacian, and where we can apply the maximum principle and all its associated tools. As we will see, this difference between Dirichlet and Navier conditions will determine the techniques that we will use in each case.

Moreover, due to the nonlinearity of the problems we will consider, we will be or we will not be able to use variational formulation depending also on the boundary conditions.

Organization, main results and conclusions.

Finally, we briefly summarize the organization of this work and the main results contained in every chapter. The thesis is conformed by five chapters, grouped in three parts.

Part I: Singular nonlocal problems on bounded domains.

This first part is concerned with several nonlocal problems on bounded domains, with the fractional Laplacian as main operator, with a common point: in the right hand side of all of them a singular term appears, although it may be of very different nature. In this part of the work, it will always hold $s \in (0,1)$.

In **Chapter 1** we study an elliptic concave-convex problem where the Hardy potential interferes with the fractional Laplacian. In particular, we will study the problem

$$(P_{\lambda,\mu})\begin{cases} (-\Delta)^s u - \lambda \dfrac{u}{|x|^{2s}} = u^p + \mu u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega\,, \end{cases} \tag{0.0.24}$$

where $0 \in \Omega$, $N > 2s$, $\mu > 0$, $0 < q < 1$, $0 < \lambda < \Lambda_{N,s}$ and $p > 1$.

First of all, by means of a comparison argument, we will see that all the solutions of this problem are singular at the origin. This fact is closely related to the behavior of the radial solutions to the homogeneous problem in $\mathbb{R}^N$, that is,

$$(-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{0.0.25}$$

what justifies the deep study of these solutions that we will perform in this chapter.

Later on, we will find a threshold $p(\lambda, s)$ such that for $1 < p < p(\lambda, s)$ we can find at least one positive solution to the problem, while for $p > p(\lambda, s)$ we will prove not only nonexistence, but also instantaneous and complete blow-up, where the mandatory singularity of the possible solutions will play a fundamental role. Nonexistence will be also seen for $\lambda > \Lambda_{N,s}$.

Indeed, in the case $1 < p < p(\lambda, s)$, by a monotonicity approach we can prove the existence of a positive solution for every $0 < \mu \leqslant M < +\infty$, where M is defined as

$$M := \sup\{\mu > 0 : \text{ problem } (P_{\lambda,\mu}) \text{ has a solution}\}.$$

As we will see, it is natural to differenciate the type of solutions depending on the range of p. If $1 < p \leqslant 2_s^* - 1$ we will consider energy solutions, while for $2_s^* - 1 < p < p(\lambda, s)$ we will work with solutions in *distributional* sense (see Definition 1.1.3). Moreover, in the case $1 \leqslant p \leqslant 2_s^* - 1$, applying variational techniques we will prove the existence of a second solution, first when μ is small enough, and finally for every $0 < \mu < M$.

The results of this first chapter can be found in [34].

Chapter 2 is devoted to analyze the parabolic counterpart to the problem in Chapter 1. Actually, we will consider the problems

$$\begin{cases} u_t + (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + f & \text{in } \Omega \times (0,T), \\ u(x,t) > 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) = u_0(x) & \text{if } x \in \Omega, \end{cases} \tag{0.0.26}$$

and

$$\begin{cases} u_t + (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + u^p + f & \text{in } \Omega \times (0,T), \\ u(x,t) > 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) = u_0(x) & \text{if } x \in \Omega, \end{cases} \tag{0.0.27}$$

where $N > 2s$, $p > 1$, and c and λ are positive constants, $0 \in \Omega$, and f and u_0 are nonnegative functions satisfying certain summability conditions.

For the first problem, we will establish precise necessary and sufficient conditions on the summability of g and u_0 in order to have solvability. Such conditions will depend on λ, through the singularity of the radial solutions of (0.0.25). Indeed, we will need to prove that our solutions behave at the origin exactly as these radial functions.

Actually, this point makes this chapter a nontrivial extension of the classical work for the heat equation by P. Baras and J. Goldstein [44], since in the nonlocal case the proof of the singularity at the origin is much more involved than in the elliptic case. The strategy will be to transform our problem into a new one,

$$\begin{cases} |x|^{-2\gamma} v_t + L_\gamma v = |x|^{-\gamma} f(x,t) & \text{in } \Omega \times (0,T), \\ v(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ v(x,0) = v_0(x) := |x|^{\gamma} u_0(x) & \text{if } x \in \Omega, \end{cases}$$

where L_γ is a weighted operator of the form

$$L_\gamma(v(x,t)) := a_{N,s}\, \text{p.v.} \int_{\mathbb{R}^N} \frac{v(x,t) - v(y,t)}{|x-y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma},$$

that appears naturally when one writes the equation for $v(x,t) := |x|^{-\gamma} u(x,t)$, being γ the power of the radial solutions to (0.0.25). Thus, the singularity of the solutions to (0.0.26) will be a consequence of a Harnack inequality for these weighted operators. Indeed, a large part of Chapter 2 is devoted to prove this result.

Attending to the semilinear problem (0.0.27), we will prove that the same solvability behavior as in the elliptic case (Chapter 1) holds here. In particular, by comparison arguments we will prove existence of at least one solution for every $1 < p < p(\lambda, s)$ (again of energy or weak type depending on p), and nonexistence and complete blow-up for $p > p(\lambda, s)$. It is worth to point out that this barrier $p(\lambda, s)$ is exactly the same as in the elliptic case.

The results contained in Chapter 2 can be found in [2].

Finally, in the last chapter of Part I, **Chapter 3**, we consider again a singular elliptic problem, but from a very different nature. In this case, the difficulty will come up from a nonlinear term that is singular at the boundary instead of at the origin. More precisely, we will study the solvability of the problem

$$(D_{\mu,\alpha,p}) \begin{cases} (-\Delta)^s u = \dfrac{f(x)}{u^\alpha} + M u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

attending to the value of α. In this case, $N > 2s$, $M \in \{0, 1\}$, $\alpha > 0$, and f is a nonnegative function. In the case $M = 0$, we will prove the existence of an energy solution if $0 < \alpha \leqslant 1$ and f has appropriate summability, and of a weak solution in the case $\alpha > 1$ and $f \in L^1(\Omega)$. The idea to find these solutions will be to work with the truncated problems and passing to the limit afterwards.

When $M = 1$, by a more involved monotonicity argument, we will also find a solution for every $p > 1$ and for every $\mu \in (0, \Upsilon)$, where

$$\Upsilon := \sup\{\mu > 0 \text{ such that problem } (D_{\mu,\alpha,p}) \text{ has a solution}\}.$$

We will see that indeed $\Upsilon < +\infty$.

To finish this chapter, in Section 3.4 we will consider the problem

$$\begin{cases} (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + \dfrac{h(x)}{u^\alpha} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with h a nonnegative funtion, and $\alpha > 0$. Notice that studying the solvability of this problem is somehow to analyze the complementary interval of powers (that is, negative exponents) in problem (0.0.24), where all the positive powers were considered.

In particular, as in $(D_{\mu,\alpha,p})$ for $M = 0$, we prove existence of energy solution for $\alpha \leqslant 1$ and h summable enough, and of weak solution if $\alpha > 1$ and $h \in L^1(\Omega)$. Moreover, in the case $\alpha < 1$, we will find a weak solution only by requiring $h \in L^1(\Omega, |x|^{-(1-\alpha)\gamma})$, where γ is again the growth at the origin of the radial solutions of (0.0.25). As this exponent suggests, to prove this result we will make use of the theory developed in Chapter 2 for the weighted spaces.

The results in Sections 3.1 - 3.3 of this chapter can be found in [33]. Section 3.4 will appear in [3].

Part II: Bifurcation results for a critical nonlocal equation in $\mathbb{R}^N$.

The second part of this work is still concerned with nonlocal problems, but now we work in the whole $\mathbb{R}^N$ instead of a bounded domain. Furthermore, we will consider an elliptic semilinear problem without singular term, but with critical growth. More precisely, throughout **Chapter 4** we will study the problem

$$(-\Delta)^s u = \varepsilon\, h\, u^q + u^p \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$, $N > 4s$, $\varepsilon > 0$ is a small parameter, $p = 2_s^* - 1$, $0 < q < p$ and h is a continuous function satisfying

$$\omega := \mathit{supp}\, h \text{ is compact} \qquad \text{and} \qquad h_+ \not\equiv 0.$$

In particular, we will prove the existence of a solution $u_{1,\varepsilon}$ to this problem, by considering it as a perturbation of the equation

$$(-\Delta)^s u = u^p \quad \text{in } \mathbb{R}^N, \qquad p = 2_s^* - 1. \tag{0.0.28}$$

Moreover, $u_{1,\varepsilon}$ will tend to one of the solutions of (0.0.28) as $\varepsilon \to 0$, that are precisely the minimizers of the Sobolev embedding. If h changes sign, we will prove the existence of a second solution, that will converge to a different minimizer when $\varepsilon \to 0$.

To obtain these existence results, we will perform a Lyapunov-Schmidt reduction, taking advantage of the variational structure of the problem.

The results of this chapter can be found in [84].

Part III: Biharmonic elliptic problems.

In the last part, that corresponds to **Chapter 5**, we will study different local problems in dimension $N = 3$.

In particular, we will consider a fourth order operator, the bilaplacian, and several nonlinear problems, whose model will be

$$\begin{cases} \Delta^2 u = S_2(D^2 u), & \text{in } \Omega \subset \mathbb{R}^N, \\ B(u) = 0, & \text{on } \partial\Omega. \end{cases} \tag{0.0.29}$$

Here $B(u)$ denotes some generic boundary conditions, and

$$S_2(D^2 u)(x) := \sum_{1 \leqslant i < j \leqslant N} \lambda_i(x) \lambda_j(x),$$

being λ_i, with $i = 1, \ldots, N$, the eigenvalues of the Hessian matrix, that is, the solutions to the equation

$$\det\big(\lambda I - D^2u(x)\big) = 0.$$

In the case of Dirichlet boundary conditions we can find an energy functional so that our problem is the associated *Euler-Lagrange* equation. Thus, by means of variational techniques we will prove existence of at least one solution to this problem.

Moreover, we will see how the addition of a term of the form $\mu|u|^{p-1}u$ determines the multiplicity of solutions: for $p < 1$ we will find two solutions when μ is small enough; for $p > 1$ we will prove the existence of at least one solution for every $\mu \geqslant 0$; and for the linear case, $p = 1$, we will prove existence of solution whenever μ is smaller than the first eigenvalue of Δ^2 in Ω.

When we are dealing with Navier boundary conditions, a variational approach is not possible, and in fact we do not know anything about the solvability of problem (0.0.29). Nevertheless, by applying the Rabinowitz bifurcation theorem, we will see that there exists an unbounded branch of solutions to the problem

$$\begin{cases} \Delta^2 u = S_2(D^2u) + \lambda u, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega,\\ \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

bifurcating from the first eigenvalue.

Finally, we will consider a last problem with Navier boundary conditions,

$$\begin{cases} \Delta^2 u = S_2(D^2u) + \mu f(x) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\\ \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this case, we will prove existence of at least one solution for $f \in L^1(\Omega)$ and μ small enough, by means of a fixed point argument.

The results contained in this chapter can be found in [100].

PART I

Singular nonlocal problems on bounded domains

Chapter 1

A semilinear elliptic problem involving the Hardy potential

Motivated by the papers [49, 85, 93], the goal of this first chapter will be to study the interplay between the Hardy potential and the solvability of a nonlocal concave-convex problem. In particular, we will analyze the existence of non trivial solutions for the problem

$$(P_{\lambda,\mu})\quad \begin{cases} (-\Delta)^s u - \lambda \dfrac{u}{|x|^{2s}} = u^p + \mu u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega\,, \end{cases}$$

where $0 \in \Omega$, $s \in (0,1)$, $N > 2s$, $\mu > 0$, $0 < q < 1$, $\lambda < \Lambda_{N,s}$ and $p > 1$. In the classical case, this type of elliptic problems where the operator interacts with the Hardy potential has a huge literature behind. Indeed, to obtain a complete analysis of the behavior of this potential in the local framework, one can consult the references [47, 48, 49, 85, 111], and also [4, 5, 6, 7, 113], among others.

Concerning the nonlocal case, there exist also many works related to the Dirichlet problem for the fractional Laplacian with semilinear perturbations. See for instance [31, 45, 53, 170], where the operator is defined by the classical spectral theory, and [154, 155, 156, 157] for the fractional Laplacian defined by (0.0.15). Moreover, in [93], M. M. Fall extends to the nonlocal case some results given by Brezis-Dupaigne-Tesei in [49], where the Hardy potential plays an important role. In particular, he analyzes in detail the case $\mu = 0$, analyzing the threshold of the power p to have solvability, by means of the extension given by L. Caffarelli and L. Silvestre in [57]. In this chapter we extend the results of this work to the case $\mu > 0$, but dealing always with the nonlocality of the operator, that is, using the definition given by the singular integral, and never the extended Laplacian.

Remark 1.0.9. *If $\mu = 0$ and $p < 2_s^* - 1$ it is possible to find a variational solution using the classical Mountain Pass Lemma introduced by A. Ambrosetti and P. Rabinowitz in [25] (see Section 1.4). However, if $p \geqslant 2_s^* - 1$, $\mu = 0$ and Ω is a starshaped domain, the only solution in $H_0^s(\Omega)$ is the trivial one. This result follows by an argument of Pohozaev type (see [95, Corollary 1.3]). This fact motivates the term u^q, $q < 1$, in our work.*

The results of this chapter can be found in [34].

1.1 Preliminaries and functional setting.

Assume first $1 < p \leqslant 2_s^* - 1$. Thus, we can introduce the following definition.

Definition 1.1.1. We say that $u \in H^s(\mathbb{R}^N)$ is an *energy* supersolution (respectively subsolution) of $(P_{\lambda,\mu})$ if $u > 0$ a.e. in Ω, $u \geqslant (\leqslant)\, 0$ a.e. on $\mathbb{R}^N \setminus \Omega$, and for every nonnegative $\varphi \in H_0^s(\Omega)$ there holds

$$\frac{a_{N,s}}{2} \iint_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy - \lambda \int_\Omega \frac{u\varphi}{|x|^{2s}}\,dx$$
$$\geqslant (\leqslant) \int_\Omega u^p \varphi\,dx + \mu \int_\Omega u^q \varphi\,dx.$$

If $u \in H_0^s(\Omega)$ is both supersolution and subsolution, we say that it is a positive energy solution.

Observe that, since $p \leqslant 2_s^* - 1$, $u^p \in L^{\frac{2N}{N+2s}}(\Omega)$, and thus the right hand side of (1.1.1) is well defined for any $\varphi \in H_0^s(\Omega)$. Moreover, in this case problem $(P_{\lambda,\mu})$ is variational in nature. In particular, we will search for critical points of the energy functional $\mathcal{J} : H_0^s(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned}\mathcal{J}(u) \;:=\; & \frac{a_{N,s}}{4} \iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ - \; & \frac{\lambda}{2}\int_\Omega \frac{(u_+)^2}{|x|^{2s}}\,dx - \frac{1}{p+1}\int_\Omega u_+^{p+1}\,dx - \frac{\mu}{q+1}\int_\Omega u_+^{q+1}\,dx,\end{aligned}$$

whose associated Euler-Lagrange equation will be the corresponding to the problem

$$\left\{\begin{aligned} (-\Delta)^s u - \lambda \frac{u_+}{|x|^{2s}} &= u_+^p + \mu u_+^q && \text{in } \Omega, \\ u &= 0 && \text{on } \mathbb{R}^N \setminus \Omega\,. \end{aligned}\right. \tag{1.1.1}$$

That is, critical points of $\mathcal{J}$ will be solutions to problem (1.1.1).

Remark 1.1.2. *Notice that if u is a nonnegative solution of problem* (1.1.1)*, then $u_+ \equiv u$ and u is also a solution to problem $(P_{\lambda,\mu})$. In particular, as we will see in Lemma 1.1.4, every solution to* (1.1.1) *is nonnegative, and therefore, critical points of $\mathcal{J}$ will provide solutions to $(P_{\lambda,\mu})$.*

When $p > 2_s^* - 1$, the problem is supercritical and we lose the variational structure. Indeed, we need to consider a weaker notion of solution. Define first the set

$$\mathcal{T} := \{\varphi : \mathbb{R}^N \to \mathbb{R} \text{ measurable, s.t. } (-\Delta)^s \varphi \in L^\infty(\Omega) \text{ and } \varphi = 0 \text{ on } \mathbb{R}^N \setminus \Omega\}. \tag{1.1.2}$$

Notice that from [150, Proposition 1.1], every $\varphi \in \mathcal{T}$ belongs to $\mathcal{C}^s(\overline{\Omega})$ as well, and therefore $\varphi \in L^\infty(\Omega)$. Thus,

Definition 1.1.3. We say that $u \in L^1(\Omega)$ is a *weak* supersolution (respectively subsolution) of $(P_{\lambda,\mu})$ if $u > 0$ a.e. in Ω, $u \geqslant (\leqslant)\, 0$ a.e. on $\mathbb{R}^N \setminus \Omega$, and it satisfies

$$\left(\frac{u}{|x|^{2s}} + u^p + u^q\right)\delta^s \in L^1(\Omega), \qquad \text{with } \ \delta(x) := dist(x, \partial\Omega), \tag{1.1.3}$$

and

$$\int_{\mathbb{R}^N} u(-\Delta)^s \varphi \, dx \geqslant (\leqslant) \int_{\Omega} \left(\lambda \frac{u}{|x|^{2s}} + u^p + \mu u^q\right) \varphi \, dx,$$

for any $\varphi \in \mathcal{T}$.

If u is both supersolution and subsolution, we say that it is a positive weak solution.

An important tool in order to build solutions by means of iterative arguments is the comparison principles, that will allow us to set an order among solutions of related problems in some specific situations. First of all, we can prove a comparison lemma for energy solutions.

Lemma 1.1.4. *Let u, $v \in H^s(\mathbb{R}^N)$ be energy solutions to the problems*

$$\begin{cases} (-\Delta)^s u = f_1 & \textit{in } \Omega, \\ \qquad\ u = g_1 & \textit{on } \mathbb{R}^N \setminus \Omega. \end{cases} \qquad \begin{cases} (-\Delta)^s v = f_2 & \textit{in } \Omega, \\ \qquad\ v = g_2 & \textit{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.1.4}$$

respectively, with f_1, $f_2 \in H^{-s}(\Omega)$ and g_1, $g_2 \in L^2(\mathbb{R}^N \setminus \Omega)$. If $f_1 \leqslant f_2$ a.e. in Ω and $g_1 \leqslant g_2$ a.e. in $\mathbb{R}^N \setminus \Omega$, then $u \leqslant v$, a.e. in $\mathbb{R}^N$.

Proof. Define the function $w := u - v$. Due to the linearity of $(-\Delta)^s$, w solves the problem

$$\begin{cases} (-\Delta)^s w = f_1 - f_2 & \text{in } \Omega, \\ \qquad\ w = g_1 - g_2 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Consider $w_+ := \max\{w, 0\}$ as a test function in the previous problem. Therefore,

$$\frac{a_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{(w(x) - w(y))(w_+(x) - w_+(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} (f_1 - f_2) w_+ \, dx \leqslant 0. \tag{1.1.5}$$

Notice that

$$\text{if} \quad w(x) \geqslant w(y), \quad \text{then} \quad w_+(x) \geqslant w_+(y).$$

Thus,

$$(w(x) - w(y))(w_+(x) - w_+(y)) \geqslant 0$$

almost everywhere in $\mathbb{R}^{2N}$. Then from (1.1.5) we deduce that, in fact,

$$(w(x) - w(y))(w_+(x) - w_+(y)) = 0, \qquad \text{a.e. in } \mathbb{R}^{2N}.$$

Therefore $w_+(x) - w_+(y) = 0$ for almost every $x, y \in \mathbb{R}^N$, that is, there exists a constant K such that $w_+(x) \equiv K$. Since $w_+ = 0$ in $\mathbb{R}^N \setminus \Omega$, we conclude that $w_+(x) = 0$ a.e. in $\mathbb{R}^N$, and consequently $w(x) \leqslant 0$. That is, $u(x) \leqslant v(x)$. □

Likewise, we can prove the corresponding comparison principle for weak solutions.

Lemma 1.1.5. *Let u, $v \in L^1(\Omega)$ be weak solutions to the problems in* (1.1.4)*, with f_1, $f_2 \in L^1(\Omega)$ and g_1, $g_2 \in L^1(\mathbb{R}^N \setminus \Omega)$. If $f_1 \leqslant f_2$ a.e. in Ω and $g_1 \leqslant g_2$ a.e. in $\mathbb{R}^N \setminus \Omega$, then $u(x) \leqslant v(x)$, a.e. in $\mathbb{R}^N$.*

Proof. Define $w := v - u$. Thus, w solves

$$\begin{cases} (-\Delta)^s w = f_2 - f_1 \geqslant 0 & \text{in } \Omega, \\ \qquad w = g_2 - g_1 \geqslant 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1.6}$$

in the weak sense. Consider now a nonnegative function $F \in \mathcal{C}_0^\infty(\Omega)$, and let φ_F be the solution to

$$\begin{cases} (-\Delta)^s \varphi_F = F \geqslant 0 & \text{in } \Omega, \\ \qquad \varphi_F = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By [150, Proposition 1.1 and Proposition 1.4], we know $\varphi_F \in \mathcal{C}^s(\overline{\Omega}) \cap \mathcal{C}^{2s+\beta}(\Omega)$, $\beta > 0$, that is, the problem is satisfied pointwise, and φ_F can be used as a test function in (1.1.6). Moreover, $\varphi_F \geqslant 0$ (see [160, Proposition 2.2.8]). Thus, testing in (1.1.6),

$$\int_\Omega wF\,dx + \int_{\mathbb{R}^N\setminus\Omega} w(-\Delta)^s \varphi_F\,dx = \int_\Omega (f_2 - f_1)\varphi_F\,dx \geqslant 0. \tag{1.1.7}$$

Since $\varphi_F \geqslant 0$ in $\mathbb{R}^N$ and $\varphi_F = 0$ on $\mathbb{R}^N \setminus \Omega$, it follows from the definition that

$$(-\Delta)^s \varphi_F(x) \leqslant 0, \qquad x \in \mathbb{R}^N \setminus \Omega,$$

and therefore, using that $w \geqslant 0$ on $\mathbb{R}^N \setminus \Omega$, we conclude from (1.1.7) that

$$\int_\Omega wF\,dx \geqslant 0.$$

Hence, $w \geqslant 0$ a.e. in Ω. □

We end this section formulating an extension of a well-known Picone identity, that in the case of regular functions and the Laplacian operator was obtained by Picone in [147] (see also [5, 15] for an extension to positive Radon measures and the p-Laplacian with $p > 1$).

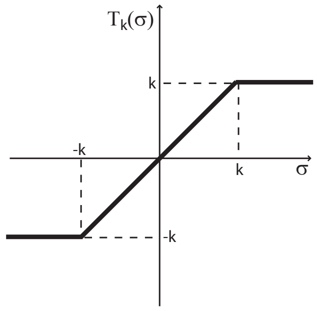
Let us precise first some notation that will be used along the whole work. Consider $k \in \mathbb{N}$ and $\sigma \in \mathbb{R}_+$. Thus, we define the functions T_k and G_k as

$$T_k(\sigma) := \max\{-k, \min\{k, \sigma\}\} \qquad \text{and} \qquad G_k(\sigma) := \sigma - T_k(\sigma). \tag{1.1.8}$$

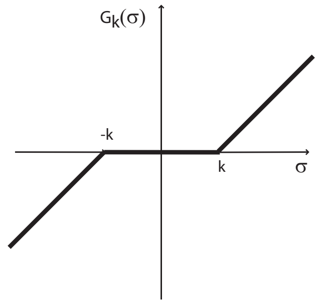
Notice that, if $u \in H_0^s(\mathbb{R}^N)$, then $T_k(u)$, $G_k(u) \in H_0^s(\mathbb{R}^N)$ (see [134, Proposition 3]).

Theorem 1.1.6. *(Picone's type Inequality).*
Consider $u, v \in H_0^s(\Omega)$, where $(-\Delta)^s u = \nu$ is a bounded Radon measure in Ω,and $u \gneq 0$. Then,

$$\int_\Omega \frac{(-\Delta)^s u}{u} v^2\,dx \leqslant \frac{a_{N,s}}{2} \|v\|^2_{H_0^s(\Omega)}. \tag{1.1.9}$$



(a) The T_k function.



(b) The G_k function.

Proof. We set, for any $k, \eta > 0$, $w := \dfrac{T_k(v)^2}{u+\eta}$, and it can be easily checked that $w \in H_0^s(\Omega)$. We want to prove that, $\forall u, v \in H_0^s(\Omega)$ and $\forall k, \eta > 0$,

$$\int (-\Delta)^s u \frac{T_k(v)^2}{u+\eta}\, dx \leqslant \frac{a_{N,s}}{2} \|T_k(v)\|^2_{H_0^s(\Omega)}, \quad \text{that is,} \quad \langle u, w\rangle_{H_0^s(\Omega)} \leqslant \|T_k(v)\|^2_{H_0^s(\Omega)}. \tag{1.1.10}$$

Once we have proved such an inequality, we obtain (1.1.9) by letting $k \to \infty$ and $\eta \to 0$, using the monotone convergence theorem.

Observe that

$$\begin{aligned}(u(x) - u(y))(w(x) - w(y)) &= \big((u(x)+\eta) - (u(y)+\eta)\big)\left(\frac{T_k(v(x))^2}{u(x)+\eta} - \frac{T_k(v(y))^2}{u(y)+\eta}\right)\\ &= T_k(v(x))^2 + T_k(v(y))^2 - T_k(v(x))^2\frac{u(y)+\eta}{u(x)+\eta} - T_k(v(y))^2\frac{u(x)+\eta}{u(y)+\eta}.\end{aligned} \tag{1.1.11}$$

Moreover,

$$T_k(v(x))^2\frac{u(y)+\eta}{u(x)+\eta} + T_k(v(y))^2\frac{u(x)+\eta}{u(y)+\eta} \geqslant 2\, T_k(v(y))T_k(v(x))$$

and hence, matching this inequality with (1.1.11), we get

$$(u(x) - u(y))(w(x) - w(y)) \leqslant \big(T_k(v(x)) - T_k(v(y))\big)^2,$$

and (1.1.10) follows. □

1.2 Radial solutions to the elliptic problem in $\mathbb{R}^N$.

The purpose of this section is to analyze the behavior in a neighborhood of the origin of the solutions to the homogeneous problem

$$(-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \tag{1.2.1}$$

in order to use this a priori information as a tool for proving the existence and nonexistence results. Indeed, we start by constructing explicit radial solutions to the equation.

Lemma 1.2.1. *Let $0 < \lambda \leqslant \Lambda_{N,s}$. Then $v_{\pm\alpha} := |x|^{-\frac{N-2s}{2}\pm\alpha}$ are pointwise solutions of the problem* (1.2.1)*, where α is given by the identity*

$$\lambda = \lambda(\alpha) = \lambda(-\alpha) = \frac{2^{2s}\,\Gamma(\frac{N+2s+2\alpha}{4})\Gamma(\frac{N+2s-2\alpha}{4})}{\Gamma(\frac{N-2s+2\alpha}{4})\Gamma(\frac{N-2s-2\alpha}{4})}. \tag{1.2.2}$$

Proof. Applying the Fourier transform of radial functions (see for instance [164, Theorem 4.1]) it yields,

$$\begin{aligned}
\mathcal{F}(v_\alpha)(\xi) &= \xi^{-\frac{N-1}{2}} \int_0^\infty (r\xi)^{\frac{1}{2}} J_{\frac{N-2}{2}}(r\xi) v_\alpha\, r^{\frac{N-1}{2}}\, dr \\
&= \xi^{-\frac{N}{2}-s-\alpha} \int_0^\infty (r\xi)^{s+\alpha} J_{\frac{N-2}{2}}(r\xi)\, d(r\xi) \\
&= 2^{\alpha+s}\frac{\Gamma(\frac{N+2s+2\alpha}{4})}{\Gamma(\frac{N-2s-2\alpha}{4})}\xi^{-\frac{N}{2}-s-\alpha},
\end{aligned}$$

where $J_{\frac{N-2}{2}}$ denotes the Bessel function of the first kind

$$J_\nu(t) := \left(\frac{t}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)}\left(\frac{t}{2}\right)^{2k}.$$

Now, we notice that

$$(-\Delta)^s v_\alpha = \mathcal{F}^{-1}(\xi^{2s}\mathcal{F}(v_\alpha)(\xi)) = 2^{\alpha+s}\frac{\Gamma(\frac{N+2s+2\alpha}{4})}{\Gamma(\frac{N-2s-2\alpha}{4})}\mathcal{F}^{-1}(\xi^{-\frac{N}{2}+s-\alpha}) = \lambda|x|^{-2s}v_\alpha,$$

with $\lambda = \lambda(\alpha)$ equal to (1.2.2). □

Remark 1.2.2. *Notice that $\lambda(\alpha) = \lambda(-\alpha) = m_\alpha m_{-\alpha}$, with $m_\alpha := 2^{\alpha+s}\frac{\Gamma(\frac{N+2s+2\alpha}{4})}{\Gamma(\frac{N-2s-2\alpha}{4})}$.*

Lemma 1.2.3. *The map*

$$\begin{array}{rcl} \lambda : [0, \frac{N-2s}{2}) & \mapsto & (0, \Lambda_{N,s}] \\ \alpha & \mapsto & \lambda(\alpha), \end{array}$$

is a bijective application. Moreover, it is strictly decreasing and

$$0 < \lambda(\alpha) \leqslant \Lambda_{N,s} \ \ \textit{if and only if} \ \ 0 \leqslant \alpha < \frac{N-2s}{2}.$$

We include here the following proof of this Lemma (see also [106, 120]).

Proof. Notice that $\lambda(\alpha)$ is a positive continuous function for $0 \leqslant \alpha < \frac{N-2s}{2}$, such that $\lambda(0) = \Lambda_{N,s}$. Thus, to prove the Lemma it is sufficient to prove that $\lambda(\alpha)$ is a strictly decreasing function. In particular, we will see that $\log \lambda(\alpha)^{-1}$ is a strictly increasing function in α, that implies the result.

Let consider the following representation of the Gamma function (see [29] for more details):

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}},$$

where γ is the Euler-Mascheroni constant (see for instance [10]). Thus,

$$\begin{aligned}\log \frac{1}{\lambda(\alpha)} &= \log \frac{1}{2^{2s}} \frac{\frac{1}{\Gamma(\frac{N+2s+2\alpha}{4})} \cdot \frac{1}{\Gamma(\frac{N+2s-2\alpha}{4})}}{\frac{1}{\Gamma(\frac{N-2s+2\alpha}{4})} \cdot \frac{1}{\Gamma(\frac{N-2s-2\alpha}{4})}} \\ &= -2s \cdot \log 2 + \log \frac{(N+2s)^2 - 4\alpha^2}{(N-2s)^2 - 4\alpha^2} + 2\gamma s + \sum_{n=1}^{\infty} \left[\log \frac{(\frac{N+4n+2s}{4n})^2 - \frac{\alpha^2}{4n^2}}{(\frac{N+4n-2s}{4n})^2 - \frac{\alpha^2}{4n^2}} - \frac{2s}{n} \right].\end{aligned}$$

Notice that the last term is a convergent series in the same way as

$$\prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}$$

is a convergent product.

We conclude just by noticing that if $a > b$, and $\zeta > 0$, then

$$\frac{a^2 - \zeta^2}{b^2 - \zeta^2}$$

is an increasing function in ζ. □

Remark 1.2.4. *Notice that we can explicitly construct two positive solutions to the homogeneous problem* (1.2.1)*. Henceforth, we denote*

$$\gamma := \frac{N-2s}{2} - \alpha \qquad \text{and} \qquad \bar{\gamma} := \frac{N-2s}{2} + \alpha, \tag{1.2.3}$$

with $0 < \gamma \leqslant \dfrac{N-2s}{2} \leqslant \bar{\gamma} < (N-2s)$*. Since* $N - 2\gamma - 2s = 2\alpha > 0$ *and* $N - 2\bar{\gamma} - 2s = -2\alpha < 0$*, then* $(-\Delta)^{s/2}(|x|^{-\gamma}) \in L^2(\Omega)$*, but* $(-\Delta)^{s/2}(|x|^{-\bar{\gamma}})$ *does not.*

As the next result will show, this information about the radial solutions of (1.2.1) is crucial to understand the behavior of the solutions to problem $(P_{\lambda,\mu})$ in a neighborhood of the origin. In particular, we can see that every solution will be, at least, as singular as $|x|^{-\gamma}$, and therefore unbounded.

Lemma 1.2.5. *Let* $0 < \lambda \leqslant \Lambda_{N,s}$ *and* $f \in L^\infty(\Omega)$*. Assume that* u *is a nonnegative function defined in* Ω *such that* $u \not\equiv 0$*,* $u \in L^1(\Omega)$*,* $\dfrac{u}{|x|^{2s}} \in L^1(\Omega)$ *and* $u = 0$ *in* $\mathbb{R}^N \setminus \Omega$*. If* u *satisfies* $(-\Delta)^s u - \lambda \dfrac{u}{|x|^{2s}} = f \geqslant 0$ *in the weak sense in* Ω*, then there exists* $\delta > 0$*, and a constant* $C = C(N, \delta)$ *such that for each ball* $B_r(0) \subset\subset \Omega$*,* $0 < r < \delta$*,*

$$u \geqslant C|x|^{-\gamma} \text{ in } B_r(0),$$

where γ *is defined in* (1.2.3)*. In particular, for* r *conveniently small we can assume that* $u > 1$ *in* $B_r(0)$*.*

Proof. Let consider ϕ the positive solution to

$$\begin{cases} (-\Delta)^s \phi = f, & \text{in } \Omega, \\ \phi = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Since $\phi \in \mathcal{C}^s(\overline{\Omega})$, applying the Strong Maximum Principle (see [160, Proposition 2.17]), for every $\delta > 0$ such that $B_\delta(0) \subset\subset \Omega$ there exists $\eta > 0$ such that $\phi \geqslant \eta$ in $B_\delta(0)$. Moreover, by Lemma 1.1.5, we know that $\phi \leqslant u$ in $\mathbb{R}^N$. Denoting $F := u - \phi$, there holds

$$\begin{cases} (-\Delta)^s F - \lambda \dfrac{F}{|x|^{2s}} = \lambda \dfrac{\phi}{|x|^{2s}} \geqslant \lambda \dfrac{\eta}{|x|^{2s}}, & \text{if } |x| \leqslant \delta, \\ F \geqslant 0, & \text{in } \mathbb{R}^N. \end{cases}$$

Consider now

$$w(x) := \begin{cases} |x|^{-\gamma} - \delta^{-\gamma}, & \text{if } |x| \leqslant \delta \\ 0, & \text{if } |x| > \delta, \end{cases}$$

where $\gamma = \dfrac{N-2s}{2} - \alpha$, and α is determined by λ (see Lemma 1.2.3). By Lemma 1.2.1,

$$(-\Delta)^s(|x|^{-\gamma}) = \lambda |x|^{-\gamma-2s} \text{ in } \mathbb{R}^N \setminus \{0\},$$

and then, for $x \in B_\delta(0)$,

$$\begin{aligned} (-\Delta)^s w(x) &= a_{N,s}\,\text{p.v.} \int_{\mathbb{R}^N} \frac{(w(x) - w(y))}{|x-y|^{N+2s}}\, dy \\ &= a_{N,s}\,\text{p.v.} \int_{\{|y| \leqslant \delta\}} \frac{|x|^{-\gamma} - |y|^{-\gamma}}{|x-y|^{N+2s}}\, dy + a_{N,s}\,\text{p.v.} \int_{\{|y| \geqslant \delta\}} \frac{|x|^{-\gamma} - \delta^{-\gamma}}{|x-y|^{N+2s}}\, dy \\ &= a_{N,s}\,\text{p.v.} \int_{\mathbb{R}^N} \frac{|x|^{-\gamma} - |y|^{-\gamma}}{|x-y|^{N+2s}}\, dy + a_{N,s}\,\text{p.v.} \int_{\{|y| \geqslant \delta\}} \frac{|y|^{-\gamma} - \delta^{-\gamma}}{|x-y|^{N+2s}}\, dy \\ &= (-\Delta)^s(|x|^{-\gamma}) + B(x) = \lambda |x|^{-\gamma-2s} + B(x). \end{aligned} \tag{1.2.4}$$

with $B(x) := a_{N,s}\,\text{p.v.} \displaystyle\int_{\{|y| \geqslant \delta\}} \frac{|y|^{-\gamma} - \delta^{-\gamma}}{|x-y|^{N+2s}}\, dy$. In order to apply comparison arguments (see Lemma 1.1.5) in $B_\delta(0)$ we have to check that w is a weak solution of (1.2.4), and thus we need $B(x) \in L^1(B_\delta(0))$. Indeed,

$$\begin{aligned} |B(x)| &\leqslant \int_{|y| \geqslant \delta} \frac{\left||y|^{-\gamma} - \delta^{-\gamma}\right|}{|x-y|^{N+2s}}\, dy = \int_{|y| \geqslant \delta} \frac{\delta^{-\gamma} - |y|^{-\gamma}}{|x-y|^{N+2s}}\, dy \\ &= \int_{|y| \geqslant \delta+1} \frac{\delta^{-\gamma} - |y|^{-\gamma}}{|x-y|^{N+2s}}\, dy + \int_{\delta \leqslant |y| \leqslant \delta+1} \frac{\delta^{-\gamma} - |y|^{-\gamma}}{|x-y|^{N+2s}}\, dy \\ &=: B_1(x) + B_2(x), \end{aligned}$$

and thus, we want to prove that $B_1, B_2 \in L^1(B_\delta(0))$. Using that $|x-y| \sim |y|$ for $|y|$ large and $|x| \leqslant \delta$, it follows

$$\begin{aligned}\int_{B_\delta(0)} B_1(x)\,dx &\leqslant \delta^{-\gamma}\int_{B_\delta(0)}\int_{|y|\geqslant\delta+1}\frac{1}{|x-y|^{N+2s}}\,dy\,dx + \int_{B_\delta(0)}\int_{|y|\geqslant\delta+1}\frac{|y|^{-\gamma}}{|x-y|^{N+2s}}\,dy\,dx\\ &\leqslant \delta^{-\gamma}\int_{B_\delta(0)}\int_{|z|\geqslant 1}\frac{1}{|z|^{N+2s}}\,dz\,dx + C\int_{B_\delta(0)}\int_{|y|\geqslant\delta+1}\frac{|y|^{-\gamma}}{|y|^{N+2s}}\,dy\,dx\\ &\leqslant \delta^{-\gamma}\int_{B_\delta(0)}\int_1^{+\infty} r^{-1-2s}\,dr\,dx + \int_{B_\delta(0)}\int_{\delta+1}^{+\infty} r^{-1-\gamma-2s}\,dr\,dx < +\infty.\end{aligned}$$

To compute $B_2(x)$, we closely follow the arguments from [101]. Indeed, set $r := |x|$ and $\rho := |y|$. Thus, $x = rx'$ and $y = \rho y'$, where $|x'| = |y'| = 1$. Hence,

$$\begin{aligned}B_2(x) &= \int_{\delta\leqslant|y|\leqslant\delta+1}\frac{\delta^{-\gamma}-|y|^{-\gamma}}{|x-y|^{N+2s}}\,dy\\ &= \int_{\delta\leqslant\rho\leqslant\delta+1}(\delta^{-\gamma}-\rho^{-\gamma})\rho^{N-1}\left(\int_{|y'|=1}\frac{dH^{N-1}(y')}{|rx'-\rho y'|^{N+2s}}\right)d\rho\\ &= \frac{1}{r^{N+2s}}\int_\delta^{\delta+1}(\delta^{-\gamma}-\rho^{-\gamma})\rho^{N-1}\left(\int_{|y'|=1}\frac{dH^{N-1}(y')}{|x'-\frac{\rho}{r}y'|^{N+2s}}\right)d\rho.\end{aligned}$$

Defining $\sigma := \frac{\rho}{r}$, we have

$$B_2(x) \quad = \quad \frac{1}{r^{N+2s}}\int_{\frac{\delta}{r}}^{\frac{\delta+1}{r}}(\delta^{-\gamma}-(\sigma r)^{-\gamma})(\sigma r)^{N-1}K(\sigma)\,r\,d\sigma,$$

with

$$K(\sigma) := \int_{|y'|=1}\frac{dH^{N-1}(y')}{|x'-\sigma y'|^{N+2s}} = 2\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}\int_0^\pi\frac{\sin^{N-2}(\theta)}{(1-2\sigma\cos(\theta)+\sigma^2)^{\frac{N+2s}{2}}}d\theta.$$

Notice that

$$\begin{aligned}B_2(x) &= \frac{1}{r^{2s}}\int_{\frac{\delta}{r}}^{\frac{\delta+1}{r}}(\delta^{-\gamma}-(\sigma r)^{-\gamma})\sigma^{N-1}K(\sigma)\,d\sigma\\ &= \frac{1}{r^{2s+\gamma}}\int_{\frac{\delta}{r}}^{\frac{\delta+1}{r}}(\delta^{-\gamma}(\sigma r)^{\gamma}-1)\sigma^{N-\gamma-1}K(\sigma)\,d\sigma.\end{aligned}$$

In order to prove that $B_2(x)$ is finite for $|x| \leqslant \delta$, we observe that, since $\delta \leqslant \rho \leqslant \delta+1$, when r tends to zero σ tends to infinity, and $K(\sigma) \simeq \dfrac{1}{\sigma^{N+2s}}$. Hence,

$$B_2(x) \leqslant \frac{C\delta^{-\gamma}}{r^{2s}}\int_{\frac{\delta}{r}}^{\frac{\delta+1}{r}}\frac{d\sigma}{\sigma^{2s+1}} - \frac{C}{r^{2s+\gamma}}\int_{\frac{\delta}{r}}^{\frac{\delta+1}{r}}\frac{d\sigma}{\sigma^{\gamma+2s+1}} = C(\delta),$$

and therefore $B_2 \in L^1(B_\delta(0))$.

Let us define now $z := cw$, with c a positive constant. By (1.2.4), since $B(x) \leqslant 0$, it yields

$$\begin{cases} (-\Delta)^s z - \lambda \dfrac{z}{|x|^{2s}} \leqslant \lambda \dfrac{c\delta^{-\gamma}}{|x|^{2s}}, & \text{if } |x| \leqslant \delta \\ z = 0, & \text{if } |x| > \delta. \end{cases}$$

It is sufficient to consider $c \leqslant \eta\delta^\gamma$ in order to conclude (by Lemma 1.1.5) that

$$z(x) \leqslant F(x) \leqslant u(x) \text{ in } \mathbb{R}^N,$$

and we obtain the singular growth of u near the origin. □

Remark 1.2.6. *A similar result will appear in [2] as a consequence of a weighted Harnarck inequality.*

Remark 1.2.7. *If $f \in L^1(\Omega)$ we can obtain the same growth at the origin, just by considering a truncation of f and comparing afterwards the solutions to the original and the truncated problem.*

Finally, to conclude this chapter we analyze the effect of the semilinear term in the behavior of the radial solutions in $\mathbb{R}^N$, that is, we study the problem

$$(-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = u^p \qquad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{1.2.5}$$

In particular, if we choose $w := A|x|^{\frac{2s-N}{2}+\alpha}$, with A a positive constant, it will be a solution to (1.2.5) if and only if

$$A\lambda(\alpha)|x|^{-2s+\frac{2s-N}{2}+\alpha} - \lambda A|x|^{-2s+\frac{2s-N}{2}+\alpha} = A^p|x|^{(\frac{2s-N}{2}+\alpha)p},$$

where

$$\lambda(\alpha) := \frac{\pi^{2s}\Gamma(\frac{N+2s+2\alpha}{4})\Gamma(\frac{N+2s-2\alpha}{4})}{\Gamma(\frac{N-2s-2\alpha}{4})\Gamma(\frac{N-2s+2\alpha}{4})}.$$

Hence, in order to preserve homogeneity, necessarily $\frac{2s-N}{2} + \alpha = \frac{-2s}{p-1}$, and in that case the equation becomes

$$\lambda(\alpha) - \lambda = A^{p-1}.$$

Since $A > 0$, we need $\lambda(\alpha) - \lambda > 0$. By Lemma 1.2.3 we can denote $\lambda = \lambda(\beta)$, with $0 \leqslant \beta < \frac{N-2s}{2}$, and since this map is decreasing, $\lambda(\alpha) - \lambda(\beta) > 0$ is equivalent to $\beta > \alpha$, that is,

$$\beta > \frac{-2s}{p-1} + \frac{N-2s}{2}.$$

Consequently, renaming β as β_λ to emphasize the dependence on λ, p must satisfy

$$p < \frac{N+2s-2\beta_\lambda}{N-2s-2\beta_\lambda} =: p(\lambda, s). \tag{1.2.6}$$

Therefore, for $p < p(\lambda, s)$ we can construct a radial solution of (1.2.5) (see [49, 93, 106], where this technique is used as well). As we will see in Section 1.3, this radial solution will allow us to find a function that will play the role of a supersolution of problem $(P_{\lambda,\mu})$, and thanks to this fact, we will be able to perform an iterative argument to construct solutions to this problem. Moreover, we will see that $p(\lambda, s)$ is an actual threshold for the existence of solution, since in Section 1.5 we will prove that in fact it is not possible to find a solution of problem $(P_{\lambda,\mu})$ for $p > p(\lambda, s)$.

1.3 Existence of minimal solutions for $1 < p < p(\lambda, s)$.

Consider now $1 < p < p(\lambda, s)$ defined in (1.2.6). We can already prove the existence results of this section.

Proposition 1.3.1. *Let M be defined by*

$$M := \sup\{\mu > 0 : \text{ problem } (P_{\lambda,\mu}) \text{ has a solution}\}. \tag{1.3.1}$$

Then $0 < M < \infty$.

Proof. The idea is to construct a subsolution and a supersolution to the problem $(P_{\lambda,\mu})$, and performing an iterative argument afterwards . For the subsolution, consider the eigenvalue problem

$$\begin{cases} (-\Delta)^s \varphi_1 = \lambda_1 \varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3.2}$$

where

$$\lambda_1 := \inf_{u \in H_0^s(\Omega),\, u \neq 0} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx}{\displaystyle\int_\Omega u^2 \, dx} \right\}$$

is the first eigenvalue of $(-\Delta)^s$ in Ω, and φ_1 the associated first eigenfunction.

By [157, Proposition 4] we can assure that $\varphi_1 \geqslant 0$ and it belongs to $H_0^s(\Omega) \cap L^\infty(\Omega)$. Hence, taking t small enough, we have that, in Ω,

$$(-\Delta)^s (t\varphi_1) = \lambda_1 t\varphi_1 \leqslant \mu (t\varphi_1)^q \leqslant \lambda \frac{t\varphi_1}{|x|^{2s} + 1} + (t\varphi_1)^p + \mu (t\varphi_1)^q.$$

Thus, $\underline{u} := t\varphi_1$ is a subsolution of $(P_{\lambda,\mu})$, both in the energy and in the weak sense. Moreover, we can build a nonnegative sequence $\{u_k\}_{k\in\mathbb{N}}$ of solutions to the iterated problems

$$\begin{cases} (-\Delta)^s u_k = \lambda \dfrac{u_{k-1}}{|x|^{2s} + \frac{1}{k}} + u_{k-1}^p + \mu u_{k-1}^q & \text{in } \Omega, \\ u_k = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,, \end{cases} \tag{1.3.3}$$

for $k \geqslant 1$ and $u_0 := \underline{u}$. Notice that, since $\underline{u} \in H_0^s(\Omega) \cap L^\infty(\Omega)$, the solutions u_k belong to $\mathcal{C}^s(\mathbb{R}^N) \cap \mathcal{C}^{2s+\beta}(\Omega)$, $\beta > 0$. In particular, $u_k \in L^\infty(\Omega)$ and they can be understood as pointwise, energy and weak solutions of (1.3.3). Moreover, by Lemma 1.1.4, we know that

$$\underline{u} \leqslant u_1 \leqslant \dots u_k \leqslant u_{k+1}, \qquad k \in \mathbb{N}.$$

The idea now is to find a function $\overline{u}$ such that $u_k \leqslant \overline{u}$ for every k. To see this, we distinguish two cases.

(i) Sub and critical case: $1 < p \leqslant 2_s^* - 1$.

Consider $w(x) := A|x|^{-\beta}$ where $A > 0$ and β is a positive real parameter that verifies

$$\beta < \frac{N-2s}{2}.$$

Since $p \leqslant 2_s^* - 1$, for this value of β it is satisfied that

$$p\beta < \beta + 2s, \qquad \text{and} \qquad \beta(p+1) < N. \tag{1.3.4}$$

By means of the Fourier transform, $(-\Delta)^s w$ can be computed explicitely, and condition (1.3.4) and an appropiate choice of A imply that

$$(-\Delta)^s w - \lambda \frac{w}{|x|^{2s}} \geqslant w^p, \quad \text{in } \Omega.$$

Let $\eta := \inf_\Omega w > 0$. For μ small enough, taking $\overline{u} := C_1 w$ with $0 < C_1 < 1$ a suitable constant such that

$$\eta^{p-q} \geqslant \mu \frac{1}{C_1^{1-q}(1 - C_1^p)}, \tag{1.3.5}$$

it follows that

$$(-\Delta)^s \overline{u} - \lambda \frac{\overline{u}}{|x|^{2s}} \geqslant \overline{u}^p + \mu \overline{u}^q \qquad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Moreover, by (1.3.4) we also have

$$\overline{u} \in L^{p+1}(\Omega) \qquad \text{and} \qquad \frac{\overline{u}^2}{|x|^{2s}} \in L^1(\Omega). \tag{1.3.6}$$

We prove by induction that $u_k \leqslant \overline{u}$ for every k. Choosing the parameter t of $\underline{u} := t\varphi_1$ small enough, it yields $\overline{u} \geqslant \underline{u} =: u_0$. Suppose the result true up to order $k-1$, that is,

$$\underline{u} \leqslant u_{j-1} \leqslant u_j \leqslant \overline{u} \quad \text{for } j \leqslant k-1 \text{ a.e. in } \Omega.$$

Then

$$(-\Delta)^s u_k = \lambda \frac{u_{k-1}}{|x|^{2s} + \frac{1}{k}} + u_{k-1}^p + \mu u_{k-1}^q \leqslant \lambda \frac{\overline{u}}{|x|^{2s}} + \overline{u}^p + \mu \overline{u}^q,$$

and

$$(-\Delta)^s(\overline{u} - u_k) \geqslant \lambda \frac{(\overline{u} - u_{k-1})}{|x|^{2s}} + (\overline{u}^p - u_{k-1}^p) + \mu(\overline{u}^q - u_{k-1}^q) \geqslant 0,$$

where this equation is understood in a pointwise sense. Moreover, since $u_k \in L^\infty(\Omega)$, there exists $r > 0$ such that by definition $\overline{u} \geqslant u_k$ in $B_r(0) \subset \Omega$. Thus, the function $z := \overline{u} - u_k$ is continuous in $\overline{\Omega \setminus B_r(0)}$ and satisfies

$$\begin{cases} (-\Delta)^s z \geqslant 0 & \text{in } \Omega \setminus B_r(0), \\ z \geqslant 0 & \text{on } B_r(0), \\ z = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Therefore, we can apply the Strong Maximum Principle (see [160, Proposition 2.17]) to conclude that $z \geqslant 0$ in $\mathbb{R}^N$, and hence

$$\underline{u} \leqslant u_1 \leqslant \ldots u_k \leqslant \ldots \leqslant \overline{u}, \qquad k \in \mathbb{N}.$$

Moreover, testing in (1.3.3) with u_k, by (1.3.6) there holds

$$\begin{aligned} \frac{a_{N,s}}{2} \|u_k\|^2_{H^s_0(\Omega)} &= \lambda \int_\Omega \frac{u_k u_{k-1}}{|x|^{2s} + \frac{1}{k}}\, dx + \int_\Omega u_k u_{k-1}^p \, dx + \mu \int_\Omega u_k u_{k-1}^q \, dx \\ &\leqslant \lambda \int_\Omega \frac{\overline{u}^2}{|x|^{2s}}\, dx + \int_\Omega \overline{u}^{p+1}\, dx + \mu \int_\Omega \overline{u}^{q+1}\, dx \\ &\leqslant C. \end{aligned} \tag{1.3.7}$$

Therefore, up to a subsequence, we know that $u_k \rightharpoonup u$ in $H^s_0(\Omega)$. In fact, by monotonicity we can pass to the limit on the right hand side of the energy formulation of problem (1.3.3) and, since $u \in H^s_0(\Omega)$ and $p \leqslant 2^*_s - 1$, the right hand side is well defined and we conclude that u is a minimal energy solution of $(P_{\lambda,\mu})$.

(ii) Supercritical case: $2^*_s - 1 < p < p(\lambda, s)$.

In this case (see [85, 93] for more details), we consider the radial function $w(x) := A|x|^{\frac{-2s}{p-1}}$, with A a positive constant so that

$$(-\Delta)^s w - \lambda \frac{w}{|x|^{2s}} = w^p \ \text{ in } \ \mathbb{R}^N \setminus \{0\}.$$

Since $p > 2^*_s - 1 > \dfrac{N}{N-2s}$,

$$w \in L^p_{loc}(\mathbb{R}^N), \qquad \text{and} \qquad \frac{w}{|x|^{2s}} \in L^1_{loc}(\mathbb{R}^N). \tag{1.3.8}$$

Again, taking $\overline{u} := C_1 w$, with $C_1 > 0$ a suitable constant (see (1.3.5)) we get that

$$\begin{cases} (-\Delta)^s \overline{u} - \lambda \dfrac{\overline{u}}{|x|^{2s}} \geqslant \overline{u}^p + \mu \overline{u}^q & \text{in } \Omega, \\ \overline{u} > 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and proceeding as in the subcritical case, one can prove that

$$\underline{u} \leqslant u_1 \leqslant \dots u_k \leqslant \dots \leqslant \overline{u}, \qquad k \in \mathbb{N},$$

where u_k is the solution to problem (1.3.3). By monotonicity and (1.3.8) we conclude that $\{u_k\}_{k\in\mathbb{N}}$ converges in $L^1(\mathbb{R}^N)$ to a weak nonnegative solution u of $(P_{\lambda,\mu})$.

Thus, for μ small enough we have built a minimal solution for the whole range of p, that is, $M > 0$. To finish the proof we need to check that $M < +\infty$. Consider first $1 < p \leqslant 2_s^* - 1$ and the eigenvalue problem with the Hardy potential given by

$$\left\{ \begin{aligned} (-\Delta)^s \phi_1 - \lambda \frac{\phi_1}{|x|^{2s}} &= \lambda_1 \phi_1 && \text{in } \Omega, \\ \phi_1 &= 0 && \text{on } \mathbb{R}^N \setminus \Omega. \end{aligned} \right. \tag{1.3.9}$$

Note that, since $\lambda < \Lambda_{N,s}$, this problem is well defined and, following the same ideas as in the proof of assertion *b*) of [156, Lemma 9], we also know that $\phi_1 \in H_0^s(\Omega)$, $\phi_1 \geqslant 0$. Suppose that u is a positive solution to problem $(P_{\lambda,\mu})$. Then taking ϕ_1 as a test function in this problem we get that

$$\begin{aligned} \frac{a_{N,s}}{2} \iint_Q \frac{(\phi_1(x) - \phi_1(y))(u(x) - u(y))}{|x-y|^{N+2s}} \, dx\, dy - \lambda \int_\Omega \frac{\phi_1 u}{|x|^{2s}} \, dx \\ = \int_\Omega u^p \phi_1 \, dx + \mu \int_\Omega u^q \phi_1 \, dx, \end{aligned}$$

and using u as a test function in (1.3.9) it follows that

$$\int_\Omega (u^p + \mu u^q) \phi_1 \, dx = \lambda_1 \int_\Omega u \phi_1 \, dx. \tag{1.3.10}$$

If $2_s^* - 1 < p < p(\lambda, s)$, we consider $\varphi_1 \geqslant 0$, solution to (1.3.2), as a test function in $(P_{\lambda,\mu})$. Then,

$$\begin{aligned} \int_\Omega u(-\Delta)^s \varphi_1 \, dx &= \int_\Omega \left(\lambda \frac{u}{|x|^{2s}} + u^p + \mu u^q \right) \varphi_1 \, dx \\ &\geqslant \int_\Omega (u^p + \mu u^q) \, \varphi_1 \, dx. \end{aligned} \tag{1.3.11}$$

Moreover, since φ_1 is a bounded energy solution, it belongs to $\mathcal{C}^s(\overline{\Omega}) \cap \mathcal{C}^{2s+\beta}(\Omega)$ (see again [150, Proposition 1.1 and Proposition 1.4]), and satisfies problem (1.3.2) pointwise. Hence from (1.3.11) we also get

$$\lambda_1 \int_\Omega u \varphi_1 \, dx \geqslant \int_\Omega (u^p + \mu u^q) \, \varphi_1 \, dx. \tag{1.3.12}$$

Since there exist structural positive constants c_0, c_1 such that

$$t^p + \mu t^q > c_0 \mu^{c_1} t, \text{ for every } t > 0,$$

we obtain from (1.3.10) and (1.3.12) that $c_0 \mu^{c_1} < \lambda_1$. Thus, necessarily μ is bounded and $M < +\infty$ for all $1 < p < p(\lambda, s)$. □

Proposition 1.3.2. *Problem $(P_{\lambda,\mu})$ has at least one positive solution for every $0 < \mu < M$. In fact, the sequence $\{u_\mu\}$ of minimal solutions is increasing with respect to μ. If $\mu = M$ the problem $(P_{\lambda,\mu})$ admits at least one weak solution.*

Proof. Since $M > 0$, we can find a solution for a value of μ as close as we want to M. Denote this value by $\overline{\mu}$ and by $u_{\overline{\mu}}$ the associated minimal solution. Then, for all $\mu < \overline{\mu}$, we get that $u_{\overline{\mu}}$ is a supersolution for the problems $(P_{\lambda,\mu})$ with $\mu < \overline{\mu}$. Furthermore, proceeding as in the proof of Proposition 1.3.1, the solution to problem (1.3.2) can be modified to become a subsolution of $(P_{\lambda,\mu})$ for every μ, less or equal than all the supersolutions. Reproducing the iterative procedure, we conclude the existence of a solution u_μ for every $\mu \in (0, \overline{\mu})$, and therefore for every $\mu \in (0, M)$. By construction, $u_\mu \leqslant u_{\overline{\mu}}$ if $\mu < \overline{\mu}$.

For the case $\mu = M$, the idea, as in [85, Proposition 2.1], consists on passing to the limit when $\mu_n \nearrow M$ on the sequence $\{u_n\} = \{u_{\mu_n}\}$, where u_{μ_n} is the minimal solution of $(P_{\lambda,\mu})$ with $\mu = \mu_n$.

Consider the solution φ_1 to the eigenvalue problem (1.3.2) as a test function in $(P_{\lambda,\mu})$. Thus,

$$\lambda_1 \int_\Omega u_n \varphi_1 \, dx = \lambda \int_\Omega \frac{u_n \varphi_1}{|x|^{2s}} \, dx + \int_\Omega u_n^p \varphi_1 \, dx + \mu_n \int_\Omega u_n^q \varphi_1 \, dx. \tag{1.3.13}$$

By Young's inequality,

$$\lambda_1 \int_\Omega u_n \varphi_1 \, dx \leqslant \lambda_1 \left(\varepsilon \int_\Omega u_n^p \varphi_1 \, dx + \frac{C(p)}{\varepsilon^{\frac{1}{p-1}}} \int_\Omega \varphi_1 \, dx \right), \qquad \varepsilon > 0,$$

and plugging this into (1.3.13) we obtain

$$\begin{aligned} \lambda \int_\Omega \frac{u_n \varphi_1}{|x|^{2s}} \, dx + (1 - \varepsilon \lambda_1) \int_\Omega u_n^p \varphi_1 \, dx + \mu_n \int_\Omega u_n^q \varphi_1 \, dx \\ \leqslant \frac{C(p)\lambda_1}{\varepsilon^{\frac{1}{p-1}}} \int_\Omega \varphi_1 \, dx \leqslant C. \end{aligned} \tag{1.3.14}$$

Moreover, by Hopf's Lemma (see [150, Lemma 3.2]), there exists $C > 0$ such that $\varphi_1(x) \geqslant C\delta^s(x)$ in Ω, where $\delta(x) := dist(x, \partial\Omega)$, and thus we can conclude from (1.3.14) that

$$\lambda \int_\Omega \frac{u_n \delta^s}{|x|^{2s}} \, dx + \int_\Omega u_n^p \delta^s \, dx + \mu_n \int_\Omega u_n^q \delta^s \, dx \leqslant C, \tag{1.3.15}$$

where C is a constant independent of n. Let now ξ_1 be the solution to the linear problem

$$\begin{cases} (-\Delta)^s \xi_1 = 1 & \text{in } \Omega, \\ \xi_1 = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Using ξ_1 as a test function of $(P_{\lambda,\mu})$, by [150, Proposition 1.1] and (1.3.15), we obtain that

$$\begin{aligned} \int_\Omega u_n \, dx &= \lambda \int_\Omega \frac{u_n \xi_1}{|x|^{2s}} \, dx + \int_\Omega u_n^p \xi_1 \, dx + \mu_n \int_\Omega u_n^q \xi_1 \, dx \\ &\leqslant C \left(\lambda \int_\Omega \frac{u_n \delta^s}{|x|^{2s}} \, dx + \int_\Omega u_n^p \delta^s \, dx + \mu_n \int_\Omega u_n^q \delta^s \, dx \right) \\ &\leqslant C, \end{aligned}$$

with $C > 0$ independent of n. Hence, using the monotonicity, we can affirm that $\{u_n\}$ converges to a limit $u_M \in L^1(\Omega)$ as $n \to \infty$. Moreover, by the uniform bound of (1.3.15) we can pass to the limit to conclude that u_M is actually a weak solution of (P_M). □

Remark 1.3.3. The results obtained in this Section can be easily translated for the case of $q = 0$, that is, considering a function f with aproppiate growth conditions instead of the concave term u^q. In particular, it could be obtained the same existence results given in [85] in the nonlocal framework.

1.4 Subcritical case: existence of at least two nontrivial solutions.

Consider $1 < p < 2_s^* - 1$ hereafter in this section.

Taking advantage of the variational structure of $(P_{\lambda,\mu})$ we will prove the existence of, at least, two positive solutions. We will use minimization to find the first solution, and the MPL to guarantee the existence of the second one. In order to use this last result, we need to check some conditions concerning the mountain pass geometry and the compactness of the functional:

Proposition 1.4.1. *There exist $\alpha > 0$ and $\beta > 0$ such that*

a) $\mathcal{J}(u) \geqslant \beta > \mathcal{J}(0)$, for any $u \in H_0^s(\Omega)$ with $\|u\|_{H_0^s(\Omega)} = \alpha$ and μ small enough.

b) There exists $u_1 \in H_0^s(\Omega)$ positive such that $\|u_1\|_{H_0^s(\Omega)} > \alpha$ and $\mathcal{J}(u_1) < \beta$.

Proof. a) Since $q + 1 < p + 1 < 2_s^*$ and $\lambda < \Lambda_{N,s}$, by the Sobolev embedding (Theorem 0.0.4) and the Hardy inequality (Theorem 0.0.6), it can be checked that

$$\mathcal{J}(u) \geqslant g\left(\|u\|_{H_0^s(\Omega)}\right),$$

where $g(t) = c_1 t^2 - c_2 t^{p+1} - \mu c_3 t^{q+1}$, for some positive constants c_1, c_2 and c_3. Then, choosing μ small enough, there exists $\alpha > 0$ such that $\beta := g(\alpha) > 0$ and therefore $\mathcal{J}(u) \geqslant \beta$ for $u \in H_0^s(\Omega)$ with $\|u\|_{H_0^s(\Omega)} = \alpha$.

b) Fix $u_0 \in H_0^s(\Omega)$ positive such that $\|u_0\|_{H_0^s(\Omega)} = 1$ and take $t > 0$. Since $p > 1$, it is clear that

$$\lim_{t\to\infty} \mathcal{J}(tu_0) = -\infty.$$

Therefore, there exists t_0 large enough, such that, defining $u_1 := t_0 u_0$, it follows that $\|u_1\|_{H_0^s(\Omega)} > \alpha$ and $\mathcal{J}(u_1) < \beta$.

□

By a similar argument, for μ small enough, we obtain that

$$\lim_{t\to 0^+} \mathcal{J}(tu_0) = 0^-. \tag{1.4.1}$$

Finally, we need to check that the functional satisfies the Palais-Smale condition. Previously we prove the next

Proposition 1.4.2. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a uniformly bounded sequence in $H_0^s(\Omega)$ such that $\mathcal{J}'(u_n) \to 0$ in $H^{-s}(\Omega)$ as $n \to \infty$. Then there exists $u \in H_0^s(\Omega)$ such that, up to a subsequence, $\|u_n - u\|_{H_0^s(\Omega)} \to 0$ when $n \to \infty$.*

Proof. Since $\|u_n\|_{H_0^s(\Omega)}$ is uniformly bounded, then there exists $C > 0$, independent of n, such that

$$\|u_n\|_*^2 := \frac{a_{N,s}}{2}\|u_n\|^2_{H_0^s(\Omega)} - \lambda \int_\Omega \frac{(u_n)_+^2}{|x|^{2s}}\,dx \leqslant \frac{a_{N,s}}{2}\|u_n\|^2_{H_0^s(\Omega)} \leqslant C.$$

Moreover, as a consequence of Theorem 0.0.6, $\|\cdot\|_*^2$ and $\|\cdot\|^2_{H_0^s(\Omega)}$ are equivalent norms, and thus there exists $u \in H_0^s(\Omega)$ such that, up to a subsequence,

$$\begin{gathered} u_n \rightharpoonup u \ \text{ in } H_0^s(\Omega) \ \text{ with the norm } \|\cdot\|_*, \\ u_n \to u \ \text{ in } L^r(\Omega), \ \text{for } 1 \leqslant r < 2_s^*, \quad \text{and } \ u_n \to u \ \text{ a.e. in } \mathbb{R}^N. \end{gathered} \tag{1.4.2}$$

Therefore, since $\mathcal{J}'(u_n) \to 0$ in $H^{-s}(\Omega)$, $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $H_0^s(\Omega)$, and $q+1 < p+1 < 2_s^*$, it follows from (1.4.2) that

$$\begin{aligned} &\lim_{n\to\infty}\left(\frac{a_{N,s}}{2}\iint_Q \frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \lambda\int_\Omega \frac{(u_n)_+^2}{|x|^{2s}}\,dx\right) \\ &\quad = \lim_{n\to\infty}\left(\mu\int_\Omega (u_n)_+^{q+1}\,dx + \int_\Omega (u_n)_+^{p+1}\,dx\right) \\ &\quad = \mu\int_\Omega u_+^{q+1}\,dx + \int_\Omega u_+^{p+1}\,dx. \end{aligned} \tag{1.4.3}$$

Similarly we obtain

$$\begin{aligned} &\lim_{n\to\infty}\left(\frac{a_{N,s}}{2}\iint_Q \frac{(u_n(x)-u_n(y))(u(x)-u(y))}{|x-y|^{N+2s}}\,dx\,dy - \lambda\int_\Omega \frac{(u_n)_+u}{|x|^{2s}}\,dx\right) \\ &\quad = \mu\int_\Omega u_+^{q+1}\,dx + \int_\Omega u_+^{p+1}\,dx. \end{aligned} \tag{1.4.4}$$

Hence, from (1.4.2), (1.4.3) and (1.4.4) we get

$$\lim_{n\to\infty}\|u_n\|_*^2 = \|u\|_*^2.$$

Consequently, by (1.4.2), we conclude that $\lim\limits_{n\to\infty}\|u_n - u\|_*^2 = 0$. Finally, by the equivalence of norms, we conclude that

$$\lim_{n\to\infty}\|u_n - u\|^2_{H_0^s(\Omega)} = 0.$$

□

Now we can prove the Palais-Smale condition. That is,

Proposition 1.4.3. *(Palais Smale condition)*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H_0^s(\Omega)$, and $c \in \mathbb{R}$ such that

$$\begin{aligned} &\mathcal{J}(u_n) \to c, \\ &\mathcal{J}'(u_n) \to 0 \quad in \quad H^{-s}(\Omega). \end{aligned} \tag{1.4.5}$$

Then, up to a subsequence, there exists u in $H_0^s(\Omega)$ such that

$$\lim_{n\to\infty} \|u_n - u\|^2_{H_0^s(\Omega)} = 0.$$

Proof. By (1.4.5) it follows that

$$\mathcal{J}(u_n) - \frac{1}{p+1}\langle \mathcal{J}'(u_n), u_n\rangle = c + o(1).$$

Hence, since $p+1 > 2 > q+1$, by Theorem 0.0.6 and Theorem 0.0.4, we obtain

$$\begin{aligned} c + o(1) &= \frac{a_{N,s}}{2}\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u_n\|^2_{H_0^s(\Omega)} - \lambda\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_\Omega \frac{(u_n)_+^2}{|x|^{2s}}\,dx \\ &\quad -\mu\Big(\frac{1}{q+1} - \frac{1}{p+1}\Big)\int_\Omega (u_n)_+^{q+1}\,dx \\ &\geqslant C_1\|u_n\|^2_{H_0^s(\Omega)} - C_2\|u_n\|^{q+1}_{H_0^s(\Omega)}, \end{aligned}$$

with C_1 and C_2 positive constants. Therefore $\|u_n\|_{H_0^s(\Omega)} \leqslant C$, and we conclude by the previous proposition. □

Now we can already state the following existence theorem.

Theorem 1.4.4. *There exists $\mu_0 > 0$ such that if $0 < \mu < \mu_0$, problem $(P_{\lambda,\mu})$ has at least two positive energy solutions.*

Proof. We construct the first solution by minimization. As we saw in Proposition 1.4.1, there exists $\alpha > 0$ such that $\mathcal{J}(u) \geqslant \beta > 0$ for all $u \in H_0^s(\Omega)$ with $\|u\|_{H_0^s(\Omega)} = \alpha$. Thus we can choose

$$\alpha_1 := \{\inf_{\alpha\in\mathbb{R}} \alpha : \mathcal{J}(u) > 0 \text{ for all } u \in H_0^s(\Omega) \text{ with } \|u\|_{H_0^s(\Omega)} = \alpha\}.$$

We know that $\alpha_1 > 0$, because the functional is negative close to the origin. We choose now $\alpha_2 > \alpha_1$ so close that $\mathcal{J}(u)$ is non decreasing for u with $\alpha_1 \leqslant \|u\|_{H_0^s(\Omega)} \leqslant \alpha_2$. We define now a smooth function τ as

$$\tau(t) := \begin{cases} 1, & t \leqslant \alpha_1, \\ 0, & t \geqslant \alpha_2, \end{cases}$$

and we consider the truncated functional

$$\begin{aligned} \mathcal{J}_2(u) := \frac{a_{N,s}}{4}\|u\|^2_{H_0^s(\Omega)} - \frac{\lambda}{2}\int_\Omega \frac{u_+^2}{|x|^{2s}}\,dx \\ - \frac{\tau(\|u\|_{H_0^s(\Omega)})}{p+1}\int_\Omega u_+^{p+1}\,dx - \frac{\mu}{q+1}\int_\Omega u_+^{q+1}\,dx. \end{aligned}$$

By definition,

$$\mathcal{J}_2(u) = \mathcal{J}(u) \quad \text{whenever} \quad \|u\|_{H_0^s(\Omega)} \leqslant \alpha_1,$$

and

$$\mathcal{J}_2(u) = \frac{a_{N,s}}{4}\|u\|^2_{H_0^s(\Omega)} - \frac{\lambda}{2}\int_\Omega \frac{u_+^2}{|x|^{2s}}\,dx - \frac{\mu}{q+1}\int_\Omega u_+^{q+1}\,dx,$$

whenever $\|u\|_{H_0^s(\Omega)} \geqslant \alpha_2$. Note that, by Theorem 0.0.4 and Theorem 0.0.6, since $q+1<2$ the functional $\mathcal{J}_2$ is coercive. The lower semicontinuity is given because $H_0^s(\Omega)$ is a Hilbert space. Therefore we can affirm that there exists a minimum of $\mathcal{J}_2$ with negative energy, that is also a minimum of $\mathcal{J}$. Then, for μ small enough, we have already found the first solution to $(P_{\lambda,\mu})$.

For the second one, as we have proved, for μ small enough the functional $\mathcal{J}$ satisfies the Mountain Pass geometry (Proposition 1.4.1), and satisfies the Palais-Smale condition (Proposition 1.4.3). Then if we consider

$$\Gamma := \{\gamma \in \mathcal{C}^0([0,1], H_0^s(\Omega)) : \ \gamma(0) = 0,\, \gamma(1) = u_1\}$$

where u_1 was found in Proposition 1.4.1, and

$$\beta_0 := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} \mathcal{J}(\gamma(t)),$$

the Mountain Pass Lemma ([25, 116]) assures the existence of a solution $u \in H_0^s(\Omega)$ satisfying

$$\mathcal{J}(u) = \beta_0 \geqslant \beta > 0,$$

with β specified in Proposition 1.4.1. Note that this solution and the one obtained before are different because the previous one had negative energy. Therefore, for μ small enough, problem $(P_{\lambda,\mu})$ has at least two solutions. □

Now we want to see that in fact problem $(P_{\lambda,\mu})$ has two solutions for every $\mu \in (0, M)$. As we said in the introduction, with this purpose we will generalize the result of S. Alama (see [14]) to check that the solution obtained in Proposition 1.3.2 is a local minimum, fact that will allow us to apply the Mountain Pass Lemma.

Proposition 1.4.5. *Let $1 \leqslant p < 2_s^* - 1$. Then for $0 < \mu < M$, where M is defined in* (1.3.1)*, problem $(P_{\lambda,\mu})$ has at least two solutions.*

Proof. Let $\mu_0 \in (0, M)$ and take μ_1 such that $\mu_0 < \mu_1 < M$. Then, by Proposition 1.3.2, we can consider w_{μ_0} and w_{μ_1} the minimal solutions to the problems (P_{λ,μ_0}) and (P_{λ,μ_1}) respectively, which satisfy $w_{\mu_0} < w_{\mu_1}$. Now we define

$$W := \{w \in H_0^s(\Omega) : \, 0 \leqslant w \leqslant w_{\mu_1}\}.$$

Since W is a closed convex set of $H_0^s(\Omega)$, we know that $\mathcal{J}_{\mu_0}$ is lower semicontinuous and bounded from below in W, and hence there exists $\underline{w} \in W$ such that $\mathcal{J}_{\mu_0}(\underline{w}) = \inf_{w\in W} \mathcal{J}_{\mu_0}(w)$. Let $w_0 \in H_0^s(\Omega)$ be a positive solution to

$$\begin{cases} (-\Delta)^s w_0 - \lambda \dfrac{w_0}{|x|^{2s}} = \mu_0 w_0^q & \text{in } \Omega, \\ \qquad\qquad w_0 = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

Note that, since $\lambda < \Lambda_{N,s}$, the existence of w_0 is given by minimization. Then, for $0 < \varepsilon << \mu_0$, we can affirm that $\mathcal{J}_{\mu_0}(\varepsilon w_0) < 0$ because the term with power $q+1$ dominates over the quadratic terms. Taking ε small enough, since $\varepsilon w_0 \in W$, we get that $\underline{w} \neq 0$ and $\mathcal{J}_{\mu_0}(\underline{w}) < 0$. Following the idea of the proof of [167, Theorem 2.4, p.17], adapted to the nonlocal framework, we conclude that $\underline{w}$ is a solution to the problem (P_{λ,μ_0}).

Hence, we have two possible cases. If $\underline{w} \neq w_{\mu_0}$, then we have finished because we have found two different solutions. Otherwise, if $\underline{w} = w_{\mu_0}$ and we prove that $\underline{w}$ is a local minimum of $\mathcal{J}_{\mu_0}$, then we obtain a second solution as a consequence of the Mountain Pass Lemma (see [25, 116]).

Therefore, assuming $\underline{w} = w_{\mu_0}$, our goal now is to prove that $\underline{w}$ is a local minimum of the functional $\mathcal{J}_{\mu_0}$. Let us argue by contradiction, that is, suppose $\underline{w}$ is not a local minimum of $\mathcal{J}_{\mu_0}$ in the space $H_0^s(\Omega)$. Then there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \in H_0^s(\Omega)$ such that

$$\|v_n - \underline{w}\|_{H_0^s(\Omega)} \to 0 \quad \text{and} \quad \mathcal{J}_{\mu_0}(v_n) < \mathcal{J}_{\mu_0}(\underline{w}). \tag{1.4.6}$$

Let w_{μ_1} be the minimal solution associated to μ_1. Define

$$w_n := (v_n - w_{\mu_1})^+,$$

and

$$z_n(x) := \begin{cases} 0, & v_n(x) \leqslant 0, \\ v_n(x), & 0 \leqslant v_n(x) \leqslant w_{\mu_1}(x), \\ w_{\mu_1}(x), & w_{\mu_1}(x) \leqslant v_n(x). \end{cases}$$

Notice that w_n and z_n belong to the energy space $H_0^s(\Omega)$ and consider the sets

$$\begin{aligned} T_n &:= \{x : \, z_n(x) = v_n(x)\}, \quad \tilde{T_n} := T_n \cap \Omega, \\ S_n &:= \{x : \, v_n(x) \geqslant w_{\mu_1}(x)\}, \quad \tilde{S_n} := S_n \cap \Omega. \end{aligned}$$

Note that

$$z_n(x) = w_{\mu_1}(x), \quad x \in S_n, \tag{1.4.7}$$

$$z_n(x) = v_n^+(x), \quad x \in S_n^c := \mathbb{R}^n \setminus S_n. \tag{1.4.8}$$

Define now

$$F_{\mu_0}(t) := \frac{1}{p+1} t_+^{p+1} + \frac{\mu_0}{q+1} t_+^{q+1}. \tag{1.4.9}$$

Thus,

$$\int_\Omega F_{\mu_0}(v_n) = \int_{\tilde{T_n}} F_{\mu_0}(v_n) + \int_{\tilde{S_n}} F_{\mu_0}(v_n) = \int_{\tilde{T_n}} F_{\mu_0}(z_n) + \int_{\tilde{S_n}} F_{\mu_0}(v_n).$$

By simplicity, let us denote

$$V_n(x,y) := \frac{(v_n(x) - v_n(y))^2}{|x-y|^{N+2s}},$$

$$V_n^+(x,y) := \frac{(v_n^+(x) - v_n^+(y))^2}{|x-y|^{N+2s}}, \quad V_n^-(x,y) := \frac{(v_n^-(x) - v_n^-(y))^2}{|x-y|^{N+2s}},$$

$$Z_n(x,y) := \frac{(z_n(x) - z_n(y))^2}{|x-y|^{N+2s}}$$

and

$$W_n(x,y) := \frac{(w_n(x) - w_n(y))^2}{|x-y|^{N+2s}}.$$

Hence, we have

$$\begin{aligned}
\iint_{\mathbb{R}^{2N}} V_n(x,y)\,dx\,dy &= \iint_{\mathbb{R}^{2N}} V_n^+(x,y)\,dx\,dy + \iint_{\mathbb{R}^{2N}} V_n^-(x,y)\,dx\,dy \\
&\quad +2\iint_{\mathbb{R}^{2N}} \frac{(v_n^+(x) - v_n^+(y))(-v_n^-(x) + v_n^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&= \iint_{\mathbb{R}^{2N}} V_n^+(x,y)\,dx\,dy + \iint_{\mathbb{R}^{2N}} V_n^-(x,y)\,dx\,dy \\
&\quad +4\iint_{\mathbb{R}^{2N}} \frac{v_n^+(x)v_n^-(y)}{|x-y|^{N+2s}}\,dx\,dy. \qquad (1.4.10)
\end{aligned}$$

It is clear also that

$$\int_\Omega \frac{v_n^2(x)}{|x|^{2s}}dx = \int_\Omega \frac{(v_n^+(x))^2}{|x|^{2s}}\,dx + \int_\Omega \frac{(v_n^-(x))^2}{|x|^{2s}}\,dx. \qquad (1.4.11)$$

Then from (1.4.9), (1.4.10) and (1.4.11) it follows

$$\begin{aligned}
\mathcal{J}_{\mu_0}(v_n) &= \frac{a_{N,s}}{4}\iint_{\mathbb{R}^{2N}} V_n(x,y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega \frac{v_n^2(x)}{|x|^{2s}}dx \\
&\quad -\int_\Omega F_{\mu_0}(v_n)(x)dx \\
&\geqslant \frac{a_{N,s}}{4}\left(\iint_{\mathbb{R}^{2N}} V_n^+(x,y)\,dx\,dy + \iint_{\mathbb{R}^{2N}} V_n^-(x,y)\,dx\,dy\right) \\
&\quad -\frac{\lambda}{2}\int_\Omega \frac{(v_n^+(x)^2)}{|x|^{2s}}\,dx - \frac{\lambda}{2}\int_\Omega \frac{(v_n^-(x))^2}{|x|^{2s}}\,dx \\
&\quad -\int_{\tilde{T}_n} F_{\mu_0}(z_n)(x)\,dx - \int_{\tilde{S}_n} F_{\mu_0}(v_n)(x)\,dx. \qquad (1.4.12)
\end{aligned}$$

From (1.4.8) we get

$$\begin{aligned}
\iint_{\mathbb{R}^{2N}} V_n^+(x,y)\,dx\,dy &= \int_{S_n}\int_{S_n} V_n^+(x,y)\,dx\,dy + \int_{S_n^c}\int_{S_n^c} Z_n(x,y)\,dx\,dy \\
&\quad +2\int_{S_n}\int_{S_n^c} V_n^+(x,y)\,dx\,dy. \qquad (1.4.13)
\end{aligned}$$

Also, since

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} Z_n(x,y)\,dx\,dy &= \int_{S_n}\int_{S_n} Z_n(x,y)\,dx\,dy + \int_{S_n^c}\int_{S_n^c} Z_n(x,y)\,dx\,dy\\ &\quad +2\int_{S_n}\int_{S_n^c} Z_n(x,y)\,dx\,dy,\end{aligned}$$

from (1.4.13) we obtain

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} V_n^+(x,y)\,dx\,dy &= \int_{S_n}\int_{S_n} V_n^+(x,y)\,dx\,dy + 2\int_{S_n}\int_{S_n^c} V_n^+(x,y)\,dx\,dy\\ &\quad +\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} Z_n(x,y)\,dx\,dy - 2\int_{S_n}\int_{S_n^c} Z_n(x,y)\,dx\,dy\\ &\quad -\int_{S_n}\int_{S_n} Z_n(x,y)\,dx\,dy.\end{aligned} \tag{1.4.14}$$

Moreover, using the same argument,

$$\begin{aligned}\int_{\Omega} \frac{(v_n^+(x))^2}{|x|^{2s}}\,dx &= \int_{\tilde{S_n}} \frac{(v_n^+(x))^2}{|x|^{2s}}\,dx + \int_{(\tilde{S_n})^c} \frac{z_n^2(x)}{|x|^{2s}}\,dx\\ &= \int_{\tilde{S_n}} \frac{(v_n^+(x))^2}{|x|^{2s}}\,dx + \int_{\Omega} \frac{z_n^2(x)}{|x|^{2s}}\,dx - \int_{\tilde{S_n}} \frac{z_n^2(x)}{|x|^{2s}}\,dx.\end{aligned} \tag{1.4.15}$$

Therefore from (1.4.12), by the Hardy inequality given in Theorem 0.0.6, (1.4.14) and (1.4.15), we get that

$$\begin{aligned}\mathcal{J}_{\mu_0}(v_n) &\geqslant C_1\|v_n^-\|^2_{H_0^s(\Omega)} + \mathcal{J}_{\mu_0}(z_n)\\ &\quad +\frac{a_{N,s}}{4}\int_{S_n}\int_{S_n} \frac{(v_n^+(x)-v_n^+(y))^2-(z_n(x)-z_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy\\ &\quad +\frac{a_{N,s}}{4}\int_{S_n}\int_{S_n^c} \frac{(v_n^+(x)-v_n^+(y))^2-(z_n(x)-z_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy\\ &\quad -\frac{\lambda}{2}\int_{\tilde{S_n}} \frac{(v_n^+(x))^2-z_n^2(x)}{|x|^{2s}}\,dx\\ &\quad +\int_{\tilde{S_n}} F_{\mu_0}(z_n)-F_{\mu_0}(v_n)\,dx,\end{aligned} \tag{1.4.16}$$

where

$$C_1 = \frac{a_{N,s}}{4}\left(1-\frac{\lambda}{\Lambda_{N,s}}\right).$$

Since $w_n(x) = v_n(x) - w_{\mu_1}(x)$ when $x \in S_n$, using (1.4.7) we obtain that

(1.4.17)
$$\begin{aligned}
&\int_{S_n}\int_{S_n} \frac{(v_n^+(x) - v_n^+(y))^2 - (z_n(x) - z_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad = \int_{S_n}\int_{S_n} \frac{(w_n(x) + w_{\mu_1}(x) - w_n(y) - w_{\mu_1}(y))^2 - (w_{\mu_1}(x) - w_{\mu_1}(y))^2}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad = \int_{S_n}\int_{S_n} \frac{(w_n(x) - w_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy \\
&\qquad + \int_{S_n}\int_{S_n} \frac{2(w_n(x) - w_n(y))(w_{\mu_1}(x) - w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\tilde{S_n}} \frac{(v_n^+(x))^2 - z_n^2(x)}{|x|^{2s}}\,dx &= \int_{\tilde{S_n}} \frac{(w_n(x) + w_{\mu_1}(x))^2 - w_{\mu_1}(x)^2}{|x|^{2s}}\,dx \\
&= \int_{\tilde{S_n}} \frac{w_n^2(x) + 2w_n w_{\mu_1}(x)}{|x|^{2s}}\,dx. \qquad (1.4.18)
\end{aligned}$$

Likewise, from (1.4.7) and (1.4.8) it follows that

$$\begin{aligned}
&\int_{S_n}\int_{S_n^c} \frac{(v_n^+(x) - v_n^+(y))^2 - (z_n(x) - z_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy \\
&= \int_{S_n}\int_{S_n^c} \frac{(v_n^+(x) - w_n(y) - w_{\mu_1}(y))^2 - (v_n^+(x) - w_{\mu_1}(y))^2}{|x-y|^{N+2s}}\,dx\,dy \qquad (1.4.19) \\
&= \int_{S_n}\int_{S_n^c} \frac{w_n^2(y) - 2w_n(y)(v_n^+(x) - w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy.
\end{aligned}$$

Furthermore, since $\operatorname{supp} w_n = S_n$, then

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} W_n(x,y)\,dx\,dy &= \int_{S_n}\int_{S_n} W_n(x,y)\,dx\,dy \\
&\quad +2\int_{S_n}\int_{S_n^c} \frac{w_n^2(y)}{|x-y|^{N+2s}}\,dx\,dy. \qquad (1.4.20)
\end{aligned}$$

Thus using (1.4.17), (1.4.18), (1.4.19) and (1.4.20), from (1.4.16) we get that

$$\begin{aligned}
\mathcal{J}_{\mu_0}(v_n) &\geqslant C_1\|v_n^-\|^2_{H_0^s(\Omega)} + \frac{a_{N,s}}{4}\|w_n\|^2_{H_0^s(\Omega)} + \mathcal{J}_{\mu_0}(z_n) \\
&\quad + \frac{a_{N,s}}{2}\int_{S_n}\int_{S_n} \frac{(w_n(x) - w_n(y))(w_{\mu_1}(x) - w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad - a_{N,s}\int_{S_n}\int_{S_n^c} \frac{w_n(y)(v_n^+(x) - w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad - \frac{\lambda}{2}\int_{\tilde{S}_n} \frac{w_n^2(x) + 2w_n w_{\mu_1}(x)}{|x|^{2s}}\,dx \\
&\quad + \int_{\tilde{S_n}} F_{\mu_0}(z_n) - F_{\mu_0}(v_n)\,dx. \qquad (1.4.21)
\end{aligned}$$

Since $v_n^+(x) \leqslant w_{\mu_1}(x)$, for $x \in S_n^c$, using that $\operatorname{supp} w_n = S_n$, from (1.4.21) it follows that

$$\begin{aligned}
\mathcal{J}_{\mu_0}(v_n) \;\geqslant\; & C_1\|v_n^-\|^2_{H_0^s(\Omega)} + \frac{a_{N,s}}{4}\|w_n\|^2_{H_0^s(\Omega)} + \mathcal{J}_{\mu_0}(z_n) \\
& + \frac{a_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{(w_n(x)-w_n(y))(w_{\mu_1}(x)-w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy \\
& - \frac{\lambda}{2}\int_{\tilde{S}_n} \frac{w_n^2(x)+2w_n w_{\mu_1}(x)}{|x|^{2s}}\,dx \\
& + \int_{\tilde{S}_n} F_{\mu_0}(z_n) - F_{\mu_0}(v_n)\,dx. \qquad (1.4.22)
\end{aligned}$$

Since w_{μ_1} is a supersolution of (P_{μ_0}), testing in that problem with the function w_n, we obtain that

$$\begin{aligned}
& \frac{a_{N,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(w_n(x)-w_n(y))(w_{\mu_1}(x)-w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy - \lambda\int_\Omega \frac{w_n w_{\mu_1}(x)}{|x|^{2s}}dx \\
& \qquad \geqslant \int_{\tilde{S}_n} w_n F'_{\mu_0}(w_{\mu_1})\,dx. \qquad (1.4.23)
\end{aligned}$$

Then from (1.4.22), Theorem 0.0.6 and (1.4.23) we have that

$$\begin{aligned}
\mathcal{J}_{\mu_0}(v_n) \;\geqslant\; & C_1\|v_n^-\|^2_{H_0^s(\Omega)} + \frac{a_{N,s}}{4}\|w_n\|^2_{H_0^s(\Omega)} - \frac{\lambda}{2}\int_{\tilde{S}_n} \frac{w_n^2(x)}{|x|^{2s}}\,dx + \mathcal{J}_{\mu_0}(z_n) \\
& + \int_{\tilde{S}_n} (F_{\mu_0}(w_{\mu_1}) - F_{\mu_0}(w_{\mu_1}+w_n) + w_n F'_{\mu_0}(w_{\mu_1}))(x)\,dx \\
\geqslant\; & \frac{a_{N,s}}{4}\|w_n\|^2_{H_0^s(\Omega)} - \frac{\lambda}{2}\int_\Omega \frac{w_n^2(x)}{|x|^{2s}}\,dx + \mathcal{J}_{\mu_0}(z_n) \\
& + \int_{\tilde{S}_n} \frac{1}{p+1}(w_{\mu_1}^{p+1} - (w_{\mu_1}+w_n)^{p+1})(x) + w_n w_{\mu_1}^p(x)\,dx \\
& + \mu_0\int_{\tilde{S}_n} \frac{(w_{\mu_1}^{q+1} - (w_{\mu_1}+w_n)^{q+1})(x)}{q+1} + w_n w_{\mu_1}^q(x)\,dx. \qquad (1.4.24)
\end{aligned}$$

Since $0 < q+1 < 2$ it follows that

$$0 \leqslant \frac{1}{q+1}[(w_{\mu_1}+w_n)^{q+1} - w_{\mu_1}^{q+1}] - w_n w_{\mu_1}^q \leqslant \frac{q}{2}\frac{w_n^2}{w_{\mu_1}^{1-q}}. \qquad (1.4.25)$$

Using that w_{μ_1} is a solution of (P_{μ_1}), since

$$|w_n(x)-w_n(y)|^2 \geqslant (w_{\mu_1}(x)-w_{\mu_1}(y))\left(\frac{w_n^2}{w_{\mu_1}}(x) - \frac{w_n^2}{w_{\mu_1}}(y)\right),$$

we obtain that

$$\begin{aligned}
\frac{a_{N,s}}{2}\|w_n\|^2_{H_0^s(\Omega)} \;\geqslant\; & \int_\Omega [\lambda\frac{w_{\mu_1}(x)}{|x|^{2s}} + \mu_1 w_{\mu_1}^q(x)]\frac{w_n^2}{w_{\mu_1}}(x)\,dx \\
\geqslant\; & \lambda\int_\Omega \frac{w_n^2(x)}{|x|^{2s}}\,dx + \mu_0\int_\Omega \frac{w_n^2}{w_{\mu_1}^{1-q}}(x)\,dx. \qquad (1.4.26)
\end{aligned}$$

Hence from (1.4.25) and (1.4.26) we get

$$
\begin{aligned}
&\frac{q}{2}\left(\frac{a_{N,s}}{2}\|w_n\|^2_{H^s_0(\Omega)} - \lambda\int_\Omega \frac{w_n^2}{|x|^{2s}}\,dx\right)\\
&\quad\geqslant \mu_0\int_{\tilde{S}_n} \frac{((w_{\mu_1}+w_n)^{q+1} - w_{\mu_1}^{q+1})(x)}{q+1} - w_n w_{\mu_1}^q(x)\,dx.
\end{aligned} \tag{1.4.27}
$$

Moreover, since $p+1>2$ we have

$$
0 \leqslant \frac{1}{p+1}[(w_{\mu_1}+w_n)^{p+1} - w_{\mu_1}^{p+1}] - w_{\mu_1}^p w_n \leqslant C(p)(w_{\mu_1}^{p-1}w_n^2 + w_n^{p+1}). \tag{1.4.28}
$$

Therefore, from (1.4.24), by (1.4.27) and (1.4.28) it follows that

$$
\begin{aligned}
\mathcal{J}_{\mu_0}(v_n) &\geqslant (1-q)\left(\frac{a_{N,s}}{4}\|w_n\|^2_{H^s_0(\Omega)} - \frac{\lambda}{2}\int_\Omega \frac{w_n^2(x)}{|x|^{2s}}\,dx\right)\\
&\quad +\mathcal{J}_{\mu_0}(z_n) + \int_{\tilde{S}_n} C(p)(-w_{\mu_1}^{p-1}w_n^2 - w_n^{p+1})(x)\,dx,\\
&\geqslant C_2\|w_n\|^2_{(\Omega)} + \mathcal{J}_{\mu_0}(z_n),\\
&\quad +\int_{\tilde{S}_n} C(p)(-w_{\mu_1}^{p-1}w_n^2 - w_n^{p+1})(x)\,dx,
\end{aligned} \tag{1.4.29}
$$

where $C_2 := (1-q)C_1 > 0$. What remains to prove now is

$$
\lim_{n\to\infty} |\tilde{S}_n| = 0. \tag{1.4.30}
$$

Let $\varepsilon, \delta > 0$, and define

$$
\begin{aligned}
A_n &= \{x\in\Omega:\ v_n(x)\geqslant w_{\mu_1}(x) \text{ and } w_{\mu_1} > \underline{w}+\delta\}\\
B_n &= \{x\in\Omega:\ v_n(x)\geqslant w_{\mu_1}(x) \text{ and } w_{\mu_1} \leqslant \underline{w}+\delta\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
0 &= |\{x\in\Omega : w_{\mu_1}(x) < \underline{w}\}| = \left|\bigcap_{j=1}^{\infty}\{x\in\Omega : w_{\mu_1}(x) < \underline{w} + \frac{1}{j}\}\right|\\
&= \lim_{j\to\infty}\left|\left\{x\in\Omega : w_{\mu_1}(x) < \underline{w} + \frac{1}{j}\right\}\right|,
\end{aligned} \tag{1.4.31}
$$

then, for j_0 large enough and $\delta < \frac{1}{j_0}$ we have

$$
|\{x\in\Omega : w_{\mu_1}(x) < \underline{w}+\delta\}| \leqslant \frac{\varepsilon}{2}.
$$

Therefore $|B_n| \leqslant \frac{\varepsilon}{2}$. Moreover, by (1.4.6),

$$
\lim_{n\to\infty}\|v_n - \underline{w}\|_{H^s_0(\Omega)} = 0,
$$

then, by Theorem 0.0.4 and the Hölder inequality, it follows that

$$\lim_{n\to\infty} \|v_n - \underline{w}\|_{L^2(\Omega)} = 0.$$

That is, for $n \geqslant n_0$ large enough we get that

$$\frac{\delta^2 \varepsilon}{2} \geqslant \int_\Omega |v_n - \underline{w}|^2\, dx \geqslant \int_{A_n} |v_n - \underline{w}|^2\, dx \geqslant \delta^2 |A_n|.$$

Therefore $|A_n| \leqslant \frac{\varepsilon}{2}$, for $n \geqslant n_0$. Since $\widetilde{S}_n \subset B_n \cup A_n$ we conclude that $|\widetilde{S}_n| \leqslant \varepsilon$ for $n \leqslant n_0$. Hence (1.4.30) holds.

Thus, by (1.4.30) and Theorem 0.0.4 we obtain

$$\int_{\tilde{S}_n} w_n^{p+1}(x) + w_{\mu_1}^{p-1} w_n^2(x)\, dx \leqslant o(1) \left(\|w_n\|^2_{H_0^s(\Omega)} + \|w_n\|^{p+1}_{H_0^s(\Omega)} \right).$$

Therefore from (1.4.29) we conclude that

$$\mathcal{J}_{\mu_0}(v_n) \geqslant C_2 \|w_n\|^2_{H_0^s(\Omega)} + \mathcal{J}_{\mu_0}(z_n) - o(1) \left(\|w_n\|^2_{H_0^s(\Omega)} + \|w_n\|^{p+1}_{H_0^s(\Omega)} \right).$$

Hence, for n large enough, since $z_n \in W$ and $\underline{w}$ was the infimum of $\mathcal{J}_{\mu_0}$ over W, this implies

$$\mathcal{J}_{\mu_0}(v_n) \geqslant \mathcal{J}_{\mu_0}(z_n) \geqslant \mathcal{J}_{\mu_0}(\underline{w}),$$

that is a contradiction with hypothesis (1.4.6). Hence $\underline{w}$ is a minimum. □

1.5 Nonexistence for $p > p(\lambda, s)$: complete blow up.

To end this chapter, it remains to study the complementary interval of powers, that is, $p \geqslant p(\lambda, s)$. As we advanced in the previous sections, we want to see that $p(\lambda, s)$ is an actual threshold, i.e., that over this value of p it is impossible to find a solution of $(P_{\lambda,\mu})$.

Theorem 1.5.1. *Let $0 < \lambda \leqslant \Lambda_{N,s}$. If $p > p(\lambda, s)$, then problem $(P_{\lambda,\mu})$ has no positive weak supersolution.*

Proof. We argue by contradiction. Suppose that there exists a positive supersolution $\overline{u} \in L^1(\Omega)$ of $(P_{\lambda,\mu})$. In particular, it will be a supersolution also if we consider the problem in a ball $B_r(0) \subset \Omega$. Thus, performing the same iterative argument as in the proof of Proposition 1.3.1, one can construct a sequence of solutions $\{u_k\}_{k\in\mathbb{N}}$ to the problems

$$\left\{ \begin{aligned} (-\Delta)^s u_k &= \lambda \frac{u_{k-1}}{|x|^{2s} + \frac{1}{k}} + u_{k-1}^p + \mu u_{k-1}^q && \text{in } B_r(0), \\ u_k &= 0 && \text{in } \mathbb{R}^N \setminus B_r(0), \end{aligned} \right. \tag{1.5.1}$$

such that

$$0 < u_k \leqslant u_{k+1} \leqslant \overline{u}, \qquad x \in B_r(0),\ k \in \mathbb{N}.$$

Moreover, every $u_k \in \mathcal{C}^s(\mathbb{R}^N) \cap \mathcal{C}^{2s+\beta}(\Omega)$, with $\beta > 0$.

Let $\phi \in \mathcal{C}_0^\infty(B_r(0))$. Thus, testing in (1.5.1) with $\dfrac{\phi^2}{u_k}$ and applying the Picone Inequality (Theorem 1.1.6),

$$\int_{B_r(0)} \frac{u_{k-1}^p}{u_k}\phi^2\,dx \leqslant \int_{B_r(0)} (-\Delta)^s u_k \frac{\phi^2}{u_k}\,dx \leqslant \frac{a_{N,s}}{2}\|\phi\|^2_{H_0^s(\Omega)}.$$

Hence, letting $k \to \infty$ and using the radial estimate in Lemma 1.2.5 we obtain

$$\begin{aligned}\frac{a_{N,s}}{2}\|\phi\|^2_{H_0^s(\Omega)} &\geqslant \int_{B_r(0)} u^{p-1}\phi^2\,dx \geqslant C\int_{B_r(0)} \frac{\phi^2}{|x|^{(p-1)\gamma}}\,dx\\ &\geqslant \frac{C}{r^{(p-1)\gamma-2s}}\int_{B_r(0)} \frac{\phi^2}{|x|^{2s}}\,dx,\end{aligned}$$

since $(p-1)\gamma > 2s$ whenever $p > p(\lambda, s)$. Choosing r small enough we obtain a contradiction with the Hardy Inequality (Theorem 0.0.6). $\square$

Moreover, we will see here that this nonexistence can be understood in the strongest sense. In particular, let us state the following definition.

Definition 1.5.2. Let u_n be the solution to the problem

$$\begin{cases} (-\Delta)^s u_n = \lambda a_n(x) T_n(u_n) + g_n(u_n) + \mu f_n(u_n) & \text{in } \Omega,\\ \qquad u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,, \end{cases}$$

where $a_n(x)$, $g_n(u)$ and $f_n(u)$ are increasing sequences of bounded functions that converge pointwise to $|x|^{-2s}$, u^p and u^q respectively, and T_n was defined in (1.1.8). We say that there is complete blow-up in problem $(P_{\lambda,\mu})$ if

$$u_n(x) \to +\infty \text{ for every } x \text{ in } \Omega.$$

Thus, the main result of this section will be the following.

Theorem 1.5.3. *Assume that $0 < \lambda \leqslant \Lambda_{N,s}$. Let $p > p(\lambda, s)$ and $\mu > 0$. Then there exists complete blow-up of the problem $(P_{\lambda,\mu})$.*

Before proving this result, we need the following auxiliary lemma that is a generalization of [47, Lemma 3.2].

Lemma 1.5.4. *Let $F(x,u) \geqslant 0$ in $L^\infty(\Omega)$, and let u be the solution of*

$$\begin{cases} (-\Delta)^s u = F(x,u) & \text{in } \Omega,\\ \qquad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.5.2}$$

Then,

$$\frac{u(x_0)}{\delta^s(x_0)} \geqslant C\int_\Omega F(x,u)\delta^s(x)\,dx, \quad x_0 \in \Omega, \tag{1.5.3}$$

where $\delta(x) = dist(x, \partial\Omega)$ and C is a positive constant depending only on Ω.

Proof. First of all we will prove (1.5.3) for points that belong to a compact set $K \subset \Omega$. Let $x_0 \in K$. By [160, Proposition 2.2.6 and Proposition 2.2.2], we have

$$u(x_0) \geqslant \int_{\mathbb{R}^N} u(x)\gamma_r(x-x_0)\,dx = \int_\Omega u(x)\gamma_r(x-x_0)\,dx > 0,$$

for any $x_0 \in \Omega$, $r \leqslant dist(x_0, \partial\Omega)$ and $\gamma_r := (-\Delta)^s\Gamma_r$ where Γ_r is a $C^{1,1}$ paraboloid that matches outside the ball $B_r(0)$ with the fundamental solution. Since u is continuous, there exist a positive constant $c > 0$ and a compact set $K \subseteq \Omega$ such that $u(x_0) \geqslant c$ for every $x_0 \in K$. That is,

$$u(x_0) \geqslant M \int_\Omega u(x)\,dx, \quad x_0 \in K, \tag{1.5.4}$$

where

$$M := c\left(\int_\Omega u(x)\,dx\right)^{-1}.$$

Consider now the problem

$$\begin{cases} (-\Delta)^s\xi_1 = 1 & \text{in } \Omega \\ \xi_1 = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,. \end{cases} \tag{1.5.5}$$

Using ξ_1 as a test function in (1.5.2), by (1.5.4) we obtain

$$u(x_0) \geqslant M \int_\Omega u(x)\,dx = M\int_\Omega F(x,u)\xi_1(x),\ x_0 \in K. \tag{1.5.6}$$

Then, by Hopf's Lemma (see [150, Lemma 3.2]),

$$u(x_0) \geqslant C \int_\Omega F(x,u)\delta^s(x)dx\,,\ x_0 \in K.$$

Moreover, since $c_1 \leqslant \delta^s(x_0) \leqslant c_2$ for $x_0 \in K$, then

$$\frac{u(x_0)}{\delta^s(x_0)} \geqslant \tilde{C}\int_\Omega F(x,u)\delta^s(x)dx\,,\ x_0 \in K. \tag{1.5.7}$$

Let now w satisfying

$$\begin{cases} (-\Delta)^s w = 0 & \text{in } \Omega \setminus K, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ w = 1 & \text{in } K. \end{cases}$$

We define

$$v(x) := \frac{u(x)}{\tilde{C}\int_\Omega F(x,u)\delta^s(x)dx},\ x \in \mathbb{R}^N.$$

Therefore

$$\begin{cases} (-\Delta)^s v \geqslant 0 & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ v \geqslant 1 & \text{in } K. \end{cases}$$

Then, by the Comparison Principle (Lemma 1.1.4) $v(x) \geqslant w(x)$ for $x \in \Omega \setminus K$. Since by Hopf's Lemma $w(x) \geqslant C\delta^s(x)$ holds, it follows that

$$\frac{u(x_0)}{\delta^s(x_0)} \geqslant C \int_\Omega F(x,u)\delta^s(x)dx\,,\ x_0 \in \Omega \setminus K. \tag{1.5.8}$$

Hence, by (1.5.7) and (1.5.8), we obtain (1.5.3). □

Now we already can prove the main result of this section.

Proof of Theorem 1.5.3. Consider the minimal solution $u_n \geqslant 0$ to the truncated problem

$$\begin{cases} (-\Delta)^s u_n = \lambda a_n(x)T_n(u_n) + g_n(u_n) + \mu f_n(u_n) & \text{in } \Omega,\\ \qquad u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,, \end{cases} \tag{1.5.9}$$

where

$$a_n(x) := T_n\left(\frac{1}{|x|^{2s}}\right), \quad g_n(u) := T_n(u_+^p) \quad \text{and} \quad f_n(u) := T_n(u_+^q),$$

with T_n defined in (1.1.8). Note that we can affirm that this minimal solution exists because, since

$$\lambda a_n(x)T_n(u_n) + g_n(u_n) + \mu f_n(u_n) \leqslant C_n,$$

we can consider, for a suitable $c > 0$, $\overline{u} := C_n\xi_1$ and $\underline{u} := c\varphi_1$ as a well ordered super and subsolution of (1.5.9) respectively. Here φ_1 is the nonnegative first eigenfunction of the fractional Laplacian defined in (1.3.2) and ξ_1 is given in (1.5.5).

We argue by contradiction. Let us suppose that

$$\int_\Omega (\lambda a_n(x)T_n(u_n) + g_n(u_n) + \mu f_n(u_n))\delta^s(x)dx \leqslant C < +\infty,\ n \in \mathbb{N}, \tag{1.5.10}$$

with C independent of n. Using ξ_1 as a test function in problem(1.5.9) we obtain that

$$\begin{aligned} \int_\Omega u_n &= \int_{\mathbb{R}^N} u_n(-\Delta)^s\xi_1\\ &= \lambda\int_\Omega a_nT_n(u_n)\xi_1 + \int_\Omega g_n(u_n)\xi_1 + \mu\int_\Omega f_n(u_n)\xi_1\\ &\leqslant C. \end{aligned}$$

Hence, up to a subsequence, $\{u_n\}_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$ to a nonnegative limit u. Then, since $a_n(x)T_n(u_n) + g_n(u_n) + \mu f_n(u_n)$ increases to $\dfrac{u}{|x|^{2s}} + u^p + \mu u^q$ in Ω, by the Monotone Convergence Theorem we can pass to the limit in (1.5.9), and by (1.5.10) we conclude that u is a nonnegative weak solution of the problem

$$\begin{cases} (-\Delta)^s u - \lambda\dfrac{u}{|x|^{2s}} = u^p + \mu u^q & \text{in } \Omega\\ \qquad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,. \end{cases}$$

But this is a contradiction with Theorem 1.5.1, and therefore

$$\lim_{n\to\infty} \int_\Omega (\lambda a_n(x)T_n(u_n) + g_n(u_n) + \mu f_n(u_n))\delta^s(x)dx = \infty.$$

Hence, we conclude applying Lemma 1.5.4. $\square$

Chapter 2

Optimal results for a parabolic problem involving the Hardy potential

In this chapter we will study the solvability of the linear problem

$$(2.0.1)\qquad \begin{cases} u_t + (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + f & \text{in } \Omega \times (0,T), \\ u(x,t) > 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) = u_0(x) & \text{if } x \in \Omega, \end{cases}$$

and of the semilinear problem

$$(2.0.2)\qquad \begin{cases} u_t + (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + u^p + f & \text{in } \Omega \times (0,T), \\ u(x,t) > 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) = u_0(x) & \text{if } x \in \Omega, \end{cases}$$

where $N > 2s$, $s \in (0,1)$, $p > 1$, and c and λ are positive constants. We assume that f and u_0 are non negative functions satisfying some hypotheses that we will precise later. Hereafter, we assume $0 \in \Omega$. Otherwise, the weight is bounded and the problems become nonsingular.

In the case $s = 1$, these problems correspond to the classical heat equation, and they have been deeply understood in the past years (see for instance [30] for (2.0.1) and [8] for (2.0.2)). For $s \in (0,1)$, the fractional setting, there exists also a large literature dealing with the case $\lambda = 0$. We refer for instance to [55, 96, 123] and the references therein. A result on the uniqueness of positive solution to the linear problem can be found in [35].

However, the case $\lambda > 0$ and $s \in (0,1)$ is quite different. Firstly, any solution to problem (2.0.1) is unbounded close to the origin, even for nice data. This fact was proved in the local case by Baras-Goldstein in [30]. Indeed, the precise rate of growth of the solutions near the origin will be the key to obtain the optimal results. In particular, it requires some sharp local estimates, that are based on a Harnack inequality for a related weighted problem.

Concerning to the semilinear problem (2.0.2), the main feature that we show in this paper is the existence of a critical exponent $p(\lambda, s)$ such that for $p > p(\lambda, s)$, there does not exist positive solution for any nontrivial nonnegative initial datum, while if $p < p(\lambda, s)$, it is possible to establish a suitable class of nonnegative data for which we can find a positive solution. Here we always assume $0 < \lambda < \Lambda_{N,s}$, where $\Lambda_{N,s}$ is the critical constant in the Hardy inequality (0.0.20). As before, the local lower estimate of the solutions close to the origin is the key to obtain the results. Furthermore, when $p > p(\lambda, s)$ the nonexistence statement is understood in the strongest possible way, that is, we prove a *complete and instantaneous blow up*, that is, if u_n are the solutions to the truncated problems, i.e., by considering $\lambda(|x|^{2s} + \frac{1}{n})^{-1}$, then $u_n(x, t) \to \infty$ as $n \to \infty$, uniformly in $\Omega \times (0, T]$.

We will see in particular that the exponent $p(\lambda, s)$ coincides with the threshold found in Chapter 1 for the elliptic case (see [34, 93]).

The results contained in Chapter 2 can be found in [2].

2.1 Preliminaries and functional setting.

Let $T_1 < T_2$. Consider the general problem

$$\begin{cases} u_t + (-\Delta)^s u = F & \text{in } \Omega \times (T_1, T_2), \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2), \\ u(x, T_1) = u_0(x) & \text{if } x \in \Omega, \end{cases} \tag{2.1.1}$$

with F and u_0 in some suitable Lebesgue space, and let us denote

$$\begin{aligned} \mathcal{T}_t := \{ \phi : \mathbb{R}^N \times [T_1, T_2] \to \mathbb{R} \text{ measurable, s.t. } & -\phi_t + (-\Delta)^s \phi = \varphi \in L^\infty(\Omega \times (T_1, T_2)), \\ & \phi = 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (T_1, T_2],\ \phi(x, T_2) = 0 \text{ in } \Omega \}. \end{aligned}$$

Notice that every $\phi \in \mathcal{T}_t$ belongs in particular to $L^\infty(\Omega \times (T_1, T_2))$ (see [134]). We define the meaning of weak solution.

Definition 2.1.1. Let $u_0 \in L^1(\Omega)$. We say that $u \in \mathcal{C}([T_1, T_2); L^1(\Omega))$ is a weak supersolution (subsolution) of problem (2.1.1) if $F \in L^1(\Omega \times (T_1, T_2))$, $u \geqslant (\leqslant)\, 0$ in $(\mathbb{R}^N \setminus \Omega) \times [T_1, T_2)$, $u(x, T_1) \geqslant (\leqslant)\, u_0(x)$ in Ω, and for all nonnegative $\phi \in \mathcal{T}_t$ we have that

$$\int_{T_1}^{T_2} \int_\Omega -\phi_t\, u\, dxdt + \int_{T_1}^{T_2} \int_{\mathbb{R}^N} u(-\Delta)^s \phi\, dxdt \geqslant (\leqslant) \int_{T_1}^{T_2} \int_\Omega F\phi\, dxdt + \int_\Omega u_0(x)\phi(x, T_1)\, dx. \tag{2.1.2}$$

If u is super and subsolution, and $u > 0$ in $\Omega \times (T_1, T_2)$, then we say that u is a positive weak solution.

Weak solutions will be considered to formulate the optimal nonexistence results, since they conform the largest class. For existence results, we will consider the classical notion of energy solutions.

Definition 2.1.2. We say that $u \in L^2(T_1, T_2; H^s(\mathbb{R}^N))$ with $u_t \in L^2(T_1, T_2; H^{-s}(\Omega))$ is an energy supersolution (subsolution) of (2.1.1) for $F \in L^2(T_1, T_2; H^{-s}(\Omega))$ and $u_0 \in L^2(\Omega)$, if $u \geqslant (\leqslant)\, 0$ in $(\mathbb{R}^N \setminus \Omega) \times [T_1, T_2)$, $u(x, T_1) \geqslant (\leqslant)\, u_0(x)$ in Ω and it satisfies

$$\begin{aligned}\int_{T_1}^{T_2}\int_\Omega u_t\varphi\, dxdt + \frac{a_{N,s}}{2}\int_{T_1}^{T_2}\iint_Q \frac{\big(u(x,t)-u(y,t)\big)\big(\varphi(x,t)-\varphi(y,t)\big)}{|x-y|^{N+2s}} dxdy\, dt \\ \geqslant (\leqslant) \int_{T_1}^{T_2}\int_\Omega F\varphi\, dxdt,\end{aligned}$$

for any nonnegative $\varphi \in L^2(T_1, T_2; H_0^s(\Omega))$, $\varphi = 0$ in $(\mathbb{R}^N \setminus \Omega) \times (T_1, T_2)$.
If $u \in L^2(T_1, T_2; H_0^s(\Omega))$ is super and subsolution, and $u > 0$ in $\Omega \times (T_1, T_2)$, we say that it is a positive energy solution.

Remark 2.1.3. *If $u \in L^2(T_1, T_2; H_0^s(\Omega))$ with $u_t \in L^2(T_1, T_2; H^{-s}(\Omega))$ is an energy solution, then by approximating with smooth functions and taking advantage of the hilbertian structure of the space, it can be checked that $u \in \mathcal{C}([T_1, T_2]; L^2(\Omega))$ similarly to the local case (see for example [92, §5.9.2, Theorem 3]). Thus, if u is an energy solution, it satisfies the identity $u(x, T_1) = u_0(x)$ in the $L^2(\Omega)$ sense.*

The existence and uniqueness of an energy solution to the problem (2.1.1) when F is in the dual space $L^2(T_1, T_2; H^{-s}(\Omega))$ can be proved by means of a direct Hilbert space approach. See the result by A. N. Milgram in [142] based on a method of Vishik in [178]. It is essentially an extension of the Lax-Milgram Theorem to parabolic problems. For the reader convenience, we include the proof here.

Theorem 2.1.4. *If $F \in L^2(T_1, T_2; H^{-s}(\Omega))$, problem* (2.1.1) *has a unique energy solution.*

Proof. Let $\mathcal{C}_*^\infty(\Omega \times [T_1, T_2])$ denote the $\mathcal{C}^\infty(\Omega \times [T_1, T_2])$ functions that vanish in $(\mathbb{R}^N \setminus \Omega) \times [T_1, T_2]$ and in $\Omega \times \{t = T_2\}$. Consider $\phi \in \mathcal{C}_*^\infty(\Omega \times [T_1, T_2])$, $u \in L^2(T_1, T_2; H_0^s(\Omega))$, and define the operator

$$L_\phi(u) := \int_{T_1}^{T_2}\int_\Omega -u\phi_t\, dxdt + \frac{a_{N,s}}{2}\int_{T_1}^{T_2}\iint_Q \frac{\big(u(x,t)-u(y,t)\big)\big(\phi(x,t)-\phi(y,t)\big)}{|x-y|^{N+2s}} dxdy\, dt.$$

Notice that u is an energy solution to (2.1.1) with $F \in L^2(T_1, T_2; H^{-s}(\Omega))$ if and only if

$$L_\phi(u) = \int_{T_1}^{T_2}\int_\Omega F\,\phi\, dxdt + \int_\Omega u(x, T_1)\phi(x, T_1)\, dx.$$

Let define the following inner product,

$$\langle \varphi, \phi\rangle_* := \frac{1}{2}\langle \varphi(x, T_1), \phi(x, T_1)\rangle_{L^2(\Omega)} + \frac{a_{N,s}}{2}\langle \varphi, \phi\rangle_{L^2(T_1,T_2;H_0^s(\Omega))},$$

and let denote $H^*(\Omega\times[T_1,T_2])$ the Hilbert space built as the completion of $\mathcal{C}^\infty_*(\Omega\times[T_1,T_2])$ with the associated norm $\|\phi\|_* := \sqrt{\langle\phi,\phi\rangle_*}$.
Applying the Hölder inequality and the Sobolev embedding,

$$\begin{aligned} |L_\phi(\varphi)| &\leqslant C(\phi)\left(\|\varphi\|_{L^2(T_1,T_2,L^2(\Omega))} + \frac{a_{N,s}}{2}\|\varphi\|_{L^2(T_1,T_2,H^s_0(\Omega))}\right) \\ &\leqslant \tilde{C}(\phi)\|\varphi\|_*. \end{aligned}$$

Therefore, L_ϕ is a linear continuous functional in $H^*(\Omega\times[T_1,T_2])$, and by the Riesz Theorem, there exists $T\phi\in H^*(\Omega\times[T_1,T_2])$ such that

$$L_\phi(\varphi) = \langle\varphi, T\phi\rangle_* \quad \text{for all } \varphi\in H^*(\Omega\times[T_1,T_2]).$$

T is a linear operator in $H^*(\Omega\times[T_1,T_2])$, and hence it is injective. Notice that

$$L_\phi(\phi) = \frac{1}{2}\int_\Omega \phi^2(x,T_1)\,dx + \frac{a_{N,s}}{2}\|\phi\|^2_{L^2(T_1,T_2;H^s_0(\Omega))} = \|\phi\|^2_*,$$

so in particular, for $\varphi=\phi$, one has $\langle\phi,T\phi\rangle_* = \|\phi\|^2_*$. Thus, by the Cauchy-Schwartz inequality,

$$\|\phi\|^2_* \leqslant \|\phi\|_*\|T\phi\|_*, \text{ i.e., } \|\phi\|_* \leqslant \|T\phi\|_*.$$

Therefore, this implies that T is bijective and its inverse T^{-1} has norm less than or equal to 1, and can be extended to the closure M of $Rank(T)$.
Define now

$$B_{u_0,F}(\phi) := \int_\Omega u(x,T_1)\phi(x,T_1)\,dx + \int_{T_1}^{T_2}\int_\Omega \phi F\,dx\,dt.$$

Denoting $\phi_0 := \phi(x,T_1)$,

$$\begin{aligned} |B_{u_0,F}(\phi)| &\leqslant \|u_0\|_{L^2(\Omega)}\|\phi_0\|_{L^2(\Omega)} + \|F\|_{L^2(T_1,T_2;L^2(\Omega))}\|\phi\|_{L^2(T_1,T_2;L^2(\Omega))} \\ &\leqslant C\left(\|u_0\|_{L^2(\Omega)} + \|F\|_{L^2(T_1,T_2;L^2(\Omega))}\right)\left(\|\phi_0\|_{L^2(\Omega)} + \|\phi\|_{L^2(T_1,T_2;H^s_0(\Omega))}\right) \\ &\leqslant \tilde{C}\|\phi\|_*, \end{aligned}$$

and moreover,

$$|B_{u_0,F}(T^{-1}\psi)| \leqslant \tilde{C}\|T^{-1}\psi\|_* \leqslant \tilde{C}\|\psi\|_*.$$

Therefore, by applying the Riesz Theorem again, there exists a unique $u\in M$ such that $B_{u_0,F}(T^{-1}\psi) = \langle\psi,u\rangle_*$ for every $\psi\in M$. Calling $\phi=T^{-1}\psi$ this means

$$B_{u_0,F}(\phi) = \langle T\phi, u\rangle_* = L_\phi(u),$$

that is,

$$\begin{aligned} \int_{T_1}^{T_2}\int_\Omega -u\phi_t\,dxdt + \frac{a_{N,s}}{2}\int_{T_1}^{T_2}\iint_Q \frac{\big(u(x,t)-u(y,t)\big)\big(\phi(x,t)-\phi(y,t)\big)}{|x-y|^{N+2s}}dxdy\,dt \\ = \int_{T_1}^{T_2}\int_\Omega F\,\phi\,dxdt + \int_\Omega u(x,T_1)\phi(x,T_1)\,dx, \end{aligned}$$

where $\phi \in L^2(T_1, T_2; H_0^s(\Omega))$ and $\phi_t \in L^2(T_1, T_2; H^{-s}(\Omega))$. Finally, by a density argument, one can conclude integrating by parts that indeed

$$u \in L^2(T_1, T_2; H_0^s(\Omega)),\ u_t \in L^2(T_1, T_2; H^{-s}(\Omega)),$$

and

$$\begin{aligned}\int_{T_1}^{T_2} \int_\Omega u_t \phi \,dxdt + \frac{a_{N,s}}{2} \int_{T_1}^{T_2} \iint_Q \frac{\big(u(x,t) - u(y,t)\big)\big(\phi(x,t) - \phi(y,t)\big)}{|x-y|^{N+2s}} dxdy\,dt \\ = \int_{T_1}^{T_2} \int_\Omega F\phi \,dxdt,\end{aligned}$$

and hence u is an energy solution of (2.1.1). $\square$

Remark 2.1.5. *Notice that by defining*

(2.1.3)
$$\begin{aligned}L_\phi(u) :=& \int_{T_1}^{T_2} \int_\Omega -u\phi_t \,dxdt + \frac{a_{N,s}}{2} \int_{T_1}^{T_2} \iint_Q \frac{\big(u(x,t) - u(y,t)\big)\big(\phi(x,t) - \phi(y,t)\big)}{|x-y|^{N+2s}} dxdy\,dt \\ &- \lambda \int_{T_1}^{T_2} \int_\Omega \frac{u\phi}{|x|^{2s}} \,dxdt,\end{aligned}$$

and

$$\langle \varphi, \phi \rangle_* := \frac{1}{2} \langle \varphi(x, T_1), \phi(x, T_1) \rangle_{L^2(\Omega)} + \frac{a_{N,s}}{2} \left(1 - \frac{\lambda}{\Lambda_{N,s}}\right) \langle \varphi, \phi \rangle_{L^2(T_1,T_2;H_0^s(\Omega))},$$

thanks to the Hardy inequality (see (0.0.20)*) one can reproduce the previous proof to assure the existence and uniqueness of an energy solution to the problem*

$$\left\{\begin{aligned} u_t + (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} &= F(x,t) && \text{in } \Omega \times (T_1, T_2), \\ u(x,t) &= 0 && \text{in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2), \\ u(x, T_1) &= u_0(x) && \text{if } x \in \Omega, \end{aligned}\right. \tag{2.1.4}$$

for $F \in L^2(T_1, T_2; H^{-s}(\Omega))$, $u_0 \in L^2(\Omega)$, and $\lambda < \Lambda_{N,s}$. For the case $\lambda = \Lambda_{N,s}$, consider the Hilbert space $H(\Omega)$ defined as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{H(\Omega)} := \frac{a_{N,s}}{2} \|u\|^2_{H_0^s(\Omega)} - \Lambda_{N,s} \int_\Omega \frac{u^2}{|x|^{2s}}\, dx. \tag{2.1.5}$$

By the improved Hardy inequality (see (0.0.21)*), we know that $H(\Omega) \subset W_0^{\tau,2}(\Omega)$ for all $s/2 < \tau < s$ and therefore, $H(\Omega)$ is compactly embedded in $L^p(\Omega)$ for all $1 \leqslant p < 2_s^*$ (see [82, Corollary 7.2]). Hence, the proof remains the same considering $L_\phi(u)$ as in* (2.1.3) *(setting $\lambda = \Lambda_{N,s}$), and defining the scalar product $\langle \cdot, \cdot \rangle_*$ as*

$$\langle \varphi, \phi \rangle_* := \frac{1}{2} \langle \varphi(x, T_1), \phi(x, T_1) \rangle_{L^2(\Omega)} + \langle \varphi, \phi \rangle_{L^2(T_1,T_2;H^s(\Omega))},$$

where the last term follows from (2.1.5)*.*

Remark 2.1.6. *Suppose that F in* (2.1.1) *belongs to $L^\infty(\Omega)$, and $u_0 \in L^2(\Omega)$. By applying [97, Corollary 4.1] to $w = u - v$, with $v \in C^s(\mathbb{R}^N)$ the solution to the elliptic problem with right hand side F, and the elliptic regularity results (see [150]) we can conclude that $u \in L^\infty(T_1, T_2; C^s(\mathbb{R}^N))$. Moreover, thanks to this regularity and to the corresponding uniqueness results, u can be understood also in viscosity sense (see [158] and [150, Remark 2.11]).*

Furthermore, attending to the results in [63, 125], if the right hand side is Hölder continuous, the solution u inherits enough regularity so that the equation in (2.1.1) *is satisfied in a pointwise sense. Thus, from now on, when we consider a problem like* (2.1.1) *with F smooth, we will be allowed to apply that the equations holds in pointwise sense.*

2.1.1 Comparison principles.

In order to study monotonicity approaches, we will need to prove comparison results for both kind of solutions.

Lemma 2.1.7. *(Energy Comparison Principle).*
Let $0 \leqslant \lambda < \Lambda_{N,s}$ and let u, $v \in L^2(T_1, T_2; H^s(\mathbb{R}^N))$ with u_t, $v_t \in L^2(T_1, T_2; H^{-s}(\Omega))$ be finite energy solutions to the problems

$$\begin{cases} u_t + (-\Delta)^s u - \lambda \dfrac{u}{|x|^{2s}} = f_1 & \text{in } \Omega \times (T_1, T_2), \\ u = g_1 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2), \\ u(x, T_1) = h_1(x) & \text{in } \Omega, \end{cases} \tag{2.1.6}$$

$$\begin{cases} v_t + (-\Delta)^s v - \lambda \dfrac{v}{|x|^{2s}} = f_2 & \text{in } \Omega \times (T_1, T_2), \\ v = g_2 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2), \\ v(x, T_1) = h_2(x) & \text{in } \Omega, \end{cases} \tag{2.1.7}$$

respectively, where f_1, $f_2 \in L^2(T_1, T_2; H^{-s}(\Omega))$, g_1, $g_2 \in L^2(T_1, T_2; L^2(\mathbb{R}^N \setminus \Omega))$ and h_1, $h_2 \in L^2(\Omega)$. If $f_1 \leqslant f_2$ a.e. in $\Omega \times (T_1, T_2)$, $g_1 \leqslant g_2$ on $(\mathbb{R}^N \setminus \Omega) \times (T_1, T_2)$ and $h_1 \leqslant h_2$ a.e. in Ω, then $u \leqslant v$ in $\mathbb{R}^N \times (T_1, T_2)$.

Proof. Let us define the function $w := u - v$. Then, since $(-\Delta)^s$ is a linear operator, w solves the problem

$$\begin{cases} w_t + (-\Delta)^s w - \lambda \dfrac{w}{|x|^{2s}} = f_1 - f_2 \leqslant 0, & \text{in } \Omega \times (T_1, T_2), \\ w = g_1 - g_2 \leqslant 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2), \\ w(x, T_1) = h_1 - h_2 \leqslant 0 & \text{in } \Omega. \end{cases}$$

Considering $w_+(x,t) := \max\{w(x,t),0\}$ as a test function in the previous problem,

(2.1.8)
$$\int_{T_1}^{T_2}\int_\Omega w_t w_+\,dx\,dt + \frac{a_{N,s}}{2}\int_{T_1}^{T_2}\iint_{\mathbb{R}^{2N}} \frac{(w(x,t)-w(y,t))(w_+(x,t)-w_+(y,t))}{|x-y|^{N+2s}}\,dxdy\,dt$$
$$-\lambda\int_{T_1}^{T_2}\int_\Omega \frac{ww_+}{|x|^{2s}}\,dx\,dt = \int_{T_1}^{T_2}\int_\Omega (f_1-f_2)w_+\,dx\,dt \leqslant 0.$$

It is easy to check that, for every t,

$$\iint_{\mathbb{R}^{2N}} \frac{(w(x,t)-w(y,t))(w_+(x,t)-w_+(y,t))}{|x-y|^{N+2s}}\,dxdy \geqslant \iint_{\mathbb{R}^{2N}} \frac{|w_+(x,t)-w_+(y,t)|^2}{|x-y|^{N+2s}}\,dxdy,$$

and thus, by the Hardy inequality,

$$\frac{a_{N,s}}{2}\int_{T_1}^{T_2}\iint_{\mathbb{R}^{2N}} \frac{(w(x,t)-w(y,t))(w_+(x,t)-w_+(y,t))}{|x-y|^{N+2s}}\,dxdy\,dt - \lambda\int_{T_1}^{T_2}\int_\Omega \frac{ww_+}{|x|^{2s}}\,dxdt \geqslant 0.$$

On the other hand, since $w_+(x,T_1)=0$ in Ω,

$$\int_{T_1}^{T_2}\int_\Omega w_t w_+\,dx\,dt = \frac{1}{2}\left(\int_\Omega (w_+)^2(x,T_2)\,dx\right) \geqslant 0.$$

Then from (2.1.8) we deduce that, in particular,

$$\frac{a_{N,s}}{2}\int_{T_1}^{T_2}\iint_{\mathbb{R}^{2N}} \frac{|w_+(x,t)-w_+(y,t)|^2}{|x-y|^{N+2s}}\,dxdy\,dt - \lambda\int_{T_1}^{T_2}\int_\Omega \frac{w_+^2}{|x|^{2s}}\,dxdt = 0,$$

and therefore, by the improved Hardy inequality (0.0.21), we can conclude $w_+(x,t)=0$ for all $(x,t)\in\mathbb{R}^N\times(T_1,T_2)$, and consequently $w(x,t)\leqslant 0$. That is, $u(x,t)\leqslant v(x,t)$ in $\mathbb{R}^N\times(T_1,T_2)$. □

Remark 2.1.8. *Notice that if $\lambda = \Lambda_{N,s}$, we can obtain the same result for u, v in $L^2(T_1,T_2;H(\Omega))$, where $H(\Omega)$ was defined in (2.1.5), only by repeating exactly this proof.*

Lemma 2.1.9. *(Weak Comparison Principle).*
Let $0\leqslant\lambda\leqslant\Lambda_{N,s}$ and let u, $v\in\mathcal{C}([T_1,T_2);L^1(\Omega))$ be weak solutions to the problems (2.1.6) and (2.1.7) respectively, with f_1, $f_2\in L^1(\Omega\times(T_1,T_2))$, g_1, $g_2\in L^1((\mathbb{R}^N\setminus\Omega)\times(T_1,T_2))$ and h_1, $h_2\in L^1(\Omega)$.

If $f_1\leqslant f_2$ a.e. in $\Omega\times(T_1,T_2)$, $g_1\leqslant g_2$ a.e. on $(\mathbb{R}^N\setminus\Omega)\times(T_1,T_2)$ and $h_1\leqslant h_2$ a.e. in $L^2(\Omega)$, then $u\leqslant v$ in $\mathbb{R}^N\times(T_1,T_2)$.

Proof. Define $w:=v-u$. Hence, w is a weak solution of

$$\begin{cases} w_t+(-\Delta)^s w-\lambda\dfrac{w}{|x|^{2s}} = f_2-f_1\geqslant 0 & \text{in } \Omega\times(T_1,T_2),\\ w = g_2-g_1\geqslant 0 & \text{in } (\mathbb{R}^N\setminus\Omega)\times[T_1,T_2),\\ w(x,T_1)=h_2-h_1\geqslant 0 & \text{in } \Omega. \end{cases}$$

Consider now $F \in \mathcal{C}_0^\infty(\Omega \times (T_1, T_2))$, $F \geqslant 0$, and the solution φ_n to the problem

$$
\begin{cases}
-(\varphi_n)_t + (-\Delta)^s \varphi_n = \lambda \dfrac{\varphi_{n-1}}{|x|^{2s} + \frac{1}{n}} + F & \text{in } \Omega \times (T_1, T_2), \\
\varphi_n = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2), \\
\varphi_n(x, T_2) = 0 & \text{in } \Omega,
\end{cases}
\tag{2.1.9}
$$

with

$$
\begin{cases}
-(\varphi_0)_t + (-\Delta)^s \varphi_0 = F & \text{in } \Omega \times (T_1, T_2), \\
\varphi_0 = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [T_1, T_2), \\
\varphi_0(x, T_2) = 0 & \text{in } \Omega.
\end{cases}
$$

Since F is smooth, the equation in (2.1.9) can be understood both pointwise and in energy sense (see Remark 2.1.6). Moreover, by Lemma 2.1.7, we know that $\varphi_n \geqslant 0$ and $\varphi_{n-1} \leqslant \varphi_n$ in $\mathbb{R}^N \times [T_1, T_2)$ for all $n \in \mathbb{N}$.

Hence, by the definition of weak solutions, and using that $w \geqslant 0$ and $(-\Delta)^s \varphi_n \leqslant 0$ on $(\mathbb{R}^N \setminus \Omega) \times [T_1, T_2)$,

$$
\begin{aligned}
&\int_{T_1}^{T_2} \int_\Omega wF \, dxdt \\
&\quad = \int_{T_1}^{T_2} \int_\Omega w(-\varphi_n)_t \, dxdt + \int_{T_1}^{T_2} \int_\Omega w(-\Delta)^s \varphi_n \, dxdt - \lambda \int_{T_1}^{T_2} \int_\Omega \frac{w\varphi_{n-1}}{|x|^{2s} + \frac{1}{n}} \, dxdt \\
&\quad \geqslant \int_{T_1}^{T_2} \int_\Omega w(-\varphi_n)_t \, dxdt + \int_{T_1}^{T_2} \int_{\mathbb{R}^{\mathbb{N}}} w(-\Delta)^s \varphi_n \, dxdt - \lambda \int_{T_1}^{T_2} \int_\Omega \frac{w\varphi_n}{|x|^{2s}} \, dxdt \\
&\quad = \int_{T_1}^{T_2} \int_\Omega (f_2 - f_1)\varphi_n \, dx \, dt + \int_\Omega w(x, T_1)\varphi_n(x, T_1) \, dx \geqslant 0,
\end{aligned}
$$

for all $F \in \mathcal{C}_0^\infty(\Omega \times (T_1, T_2))$, $F \geqslant 0$. Thus, $w \geqslant 0$ in $\mathbb{R}^N \times (T_1, T_2)$, and therefore $u \leqslant v$ in $\mathbb{R}^N \times (T_1, T_2)$. $\square$

As a consequence, we obtain the uniqueness of solution.

Corollary 2.1.10. *(Uniqueness for the linear problem).*
Let suppose $F \in L^1(\Omega \times (0, T))$. Then problem (2.1.4) *has at most one nontrivial weak solution.*

Finally, consider the problem

$$
\begin{cases}
u_t + (-\Delta)^s u = 0 & \text{in } \Omega \times (0, T), \\
u(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\
u(x, 0) = u_0(x) \gneq 0 & \text{if } x \in \Omega.
\end{cases}
\tag{2.1.10}
$$

We enunciate a *Weak Harnack Inequality*, proved by M. Felsinger and M. Kassmann in [96, Theorem 1.1] in a more general setting, that we will use throughout the paper.

Theorem 2.1.11. *(Weak Harnack Inequality).*
If u is a non negative supersolution of (2.1.10) in $\Omega \times (0,T)$, then there exists $r > 0$ and a positive constant $C = C(N, s, r, t_0, \beta)$ such that

$$\iint_{R^-} u(x,t)\,dx\,dt \leqslant C\big(\operatorname*{ess\,inf}_{R^+} u\big),$$

where $R^- := B_r(0) \times (t_0 - \frac{3}{4}\beta, t_0 - \frac{1}{4}\beta)$ and $R^+ := B_r(0) \times (t_0 + \frac{1}{4}\beta, t_0 + \frac{3}{4}\beta)$.

As a consequence of this lemma, we can formulate the *strong maximum principle.*

Theorem 2.1.12. *(Strong Maximum Principle).*
If u is a non negative supersolution of (2.1.10), then $u(x,t) > 0$ in $\Omega \times (0,T)$.

2.2 Weighted nonlocal operators. Weak Harnack inequality.

Consider γ defined in (1.2.3). Frank, Lieb and Seiringer proved in [106, Proposition 4.1] the following representation result.

Lemma 2.2.1. *(Ground State Representation)*
Let $0 < \gamma < \frac{N-2s}{2}$. If $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ and $\bar{\phi}(x) := |x|^\gamma \phi(x)$, then

$$\begin{aligned}\int_{\mathbb{R}^N} |\xi|^{2s}|\hat{\phi}(\xi)|^2\,d\xi - (\Lambda_{N,s}+\Phi_{N,s}(\gamma)) \int_{\mathbb{R}^N} |x|^{-2s}|\phi(x)|^2\,dx \\ = \frac{a_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{|\bar{\phi}(x) - \bar{\phi}(y)|^2}{|x-y|^{N+2s}} \frac{dx}{|x|^\gamma} \frac{dy}{|y|^\gamma},\end{aligned} \tag{2.2.1}$$

where $\hat{\phi} := \mathcal{F}(\phi)$ and

$$\Phi_{N,s}(\gamma) := 2^{2s}\left(\frac{\Gamma\big(\frac{\gamma+2s}{2}\big)\Gamma\big(\frac{N-\gamma}{2}\big)}{\Gamma\big(\frac{N-\gamma-2s}{2}\big)\Gamma\big(\frac{\gamma}{2}\big)} - \frac{\Gamma^2\big(\frac{N+2s}{4}\big)}{\Gamma^2\big(\frac{N-2s}{4}\big)} \right). \tag{2.2.2}$$

A relevant fact for us is the following result.

Proposition 2.2.2. *Consider the function*

$$\begin{aligned}\Psi_{N,s} : [0, \tfrac{N-2s}{2}] \quad &\to \quad [0, \Lambda_{N,s}] \\ \gamma \quad &\to \quad \Psi_{N,s}(\gamma) := \Lambda_{N,s} + \Phi_{N,s}(\gamma),\end{aligned}$$

where $\Phi_{N,s}$ is defined by (2.2.2). Then $\Psi_{N,s}$ is strictly increasing and surjective.

Notice that, following (1.2.2), $\lambda(\alpha) = \Psi_{N,s}\left(\dfrac{N-2s}{2} - \alpha\right)$, and therefore, by Lemma 1.2.3, for any $0 < \lambda \leqslant \Lambda_{N,s}$, there exists $\alpha \in [0, \frac{N-2s}{2})$, such that

$$\lambda = \lambda(\alpha) = \frac{2^{2s}\,\Gamma(\frac{N+2s+2\alpha}{4})\Gamma(\frac{N+2s-2\alpha}{4})}{\Gamma(\frac{N-2s+2\alpha}{4})\Gamma(\frac{N-2s-2\alpha}{4})},$$

and the Proposition follows.

Suppose that u is an energy solution of problem (2.0.1). Taking

$$0 < \gamma = \frac{N-2s}{2} - \alpha < \frac{N-2s}{2},$$

by Lemma 2.2.1 we can write the energy as

$$\begin{aligned}\int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}(\xi,t)|^2\,d\xi - \lambda(\alpha)\int_{\mathbb{R}^N} |x|^{-2s}|u(x,t)|^2\,dx \\ = \frac{a_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{|v(x,t)-v(y,t)|^2}{|x-y|^{N+2s}}\frac{dx}{|x|^\gamma}\frac{dy}{|y|^\gamma},\end{aligned} \tag{2.2.3}$$

where $v(x,t) := |x|^\gamma u(x,t)$ and $\hat{u} := \mathcal{F}(u)$. Hence, by (2.0.1) and (2.2.3), the Euler-Lagrange equation associated to (2.2.3) is

$$(-\Delta)^s u - \lambda\frac{u}{|x|^{2s}} = |x|^\gamma L_\gamma v(x,t),$$

where

$$L_\gamma(v(x,t)) := a_{N,s}\ \text{p.v.}\int_{\mathbb{R}^N}(v(x,t)-v(y,t))K(x,y)dy, \tag{2.2.4}$$

and

$$K(x,y) = \frac{1}{|x|^\gamma}\frac{1}{|y|^\gamma}\frac{1}{|x-y|^{N+2s}}, \quad 0<\gamma=\frac{N-2s}{2}-\alpha<\frac{N-2s}{2}. \tag{2.2.5}$$

Thus we conclude that if u is an energy solution of (2.0.1), then v solves the parabolic problem

$$\begin{cases} |x|^{-2\gamma}v_t + L_\gamma v = |x|^{-\gamma}f(x,t) & \text{in } \Omega\times(0,T), \\ v(x,t) = 0 & \text{in } (\mathbb{R}^N\setminus\Omega)\times[0,T), \\ v(x,0) = v_0(x) := |x|^\gamma u_0(x) & \text{if } x\in\Omega. \end{cases} \tag{2.2.6}$$

Therefore, in order to analyze the behavior of u near the origin, we have to deal with the same question for v. That is, we need to prove that the weighted operator

$$|x|^{-2\gamma}v_t - L_\gamma v,$$

satisfies a suitable weak Harnack inequality. In the local case, this kind of result can be obtained as a consequence of some results by Chiarenza-Frasca, Chiarenza-Serapioni and Gutierrez-Wheden (see [66, 67, 117] and the references therein). By simplicity, we work with this problem over the interval $(0,T)$, but all the results along this section apply for any interval of time (T_1,T_2), with $-\infty < T_1 < T_2 < +\infty$.

Before stating the weak Harnack inequality for this weighted operator, let us precise the natural functional framework associated to the new problem (2.2.6). For simplicity of typing we denote

$$d\mu := \frac{dx}{|x|^{2\gamma}}, \quad \text{and} \quad d\nu := K(x,y)dxdy, \tag{2.2.7}$$

with K defined in (2.2.5).

Let $\Omega \subseteq \mathbb{R}^N$. We define the weighted Sobolev space $Y^{s,\gamma}(\Omega)$ as

$$Y^{s,\gamma}(\Omega) := \Big\{\phi \in L^2(\Omega, |x|^{-\gamma}) : \iint_{\Omega\times\Omega} (\phi(x) - \phi(y))^2 d\nu < +\infty\Big\}. \tag{2.2.8}$$

It is clear that $Y^{s,\gamma}(\Omega)$ is a Hilbert space endowed with the norm

$$\|\phi\|_{Y^{s,\gamma}(\Omega)} := \Big(\int_\Omega |\phi(x)|^2 d\mu + \iint_{\Omega\times\Omega} (\phi(x) - \phi(y))^2 d\nu\Big)^{\frac{1}{2}}, \tag{2.2.9}$$

and we define the space $Y_0^{s,\gamma}(\Omega)$ as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to this norm. In particular, we denote

$$|||\phi|||_{Y_0^{s,\gamma}(\Omega)} := \Big(\iint_{\Omega\times\Omega} (\phi(x) - \phi(y)|^2 d\nu\Big)^{\frac{1}{2}}.$$

If Ω is bounded, the norms $|||\cdot|||_{Y_0^{s,\gamma}(\Omega)}$ and $\|\cdot\|_{Y^{s,\gamma}(\Omega)}$ are equivalent (see Theorem 2.2.7 for more details). If $\Omega = \mathbb{R}^N$, using the definition of L_γ, we obtain that for every $w_1, w_2 \in Y_0^{s,\gamma}(\mathbb{R}^N)$,

$$\langle L_\gamma(w_1), w_2\rangle = \frac{a_{N,s}}{2} \iint_{\mathbb{R}^{2N}} (w_1(x) - w_1(y))(w_2(x) - w_2(y)) d\nu. \tag{2.2.10}$$

Let us begin with the following definition.

Definition 2.2.3. Let $v \in L^2(0,T; Y^{s,\gamma}(\mathbb{R}^N)) \cap \mathcal{C}([0,T); L^2(\mathbb{R}^N, |x|^{-\gamma}))$. We say that v is a supersolution (subsolution) to problem (2.2.6) if $v \geqslant (\leqslant)\, 0$ on $(\mathbb{R}^N \setminus \Omega) \times [0,T)$, $v(x,0) \geqslant (\leqslant)\, v_0(x)$ and for every $\Omega_1 \subset\subset \Omega$ and every $[t_1,t_2] \subset (0,T)$ we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\Omega_1} -\varphi_t v\, d\mu\, dt + &\frac{a_{N,s}}{2}\int_{t_1}^{t_2}\int_{\tilde{Q}} (v(x,t) - v(y,t))(\varphi(x,t) - \varphi(y,t)) d\nu\, dt \\ \geqslant (\leqslant) \int_{t_1}^{t_2}\int_{\Omega_1} f\varphi\, |x|^{-\gamma} dx\, dt &+ \int_{\Omega_1} \varphi(x,t_1) v(x,t_1) d\mu - \int_{\Omega_1} \varphi(x,t_2) v(x,t_2) d\mu,\end{aligned}$$

for any nonnegative $\varphi \in L^2(t_1,t_2; Y_0^{s,\gamma}(\Omega_1))$ such that $\varphi_t \in L^2(t_1,t_2; Y^{-s,\gamma}(\Omega_1))$. Here $\tilde{Q}$ denotes $\tilde{Q} := \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega_1 \times \mathcal{C}\Omega_1)$.

We say that $v \in L^2(0,T; Y_0^{s,\gamma}(\Omega)) \cap \mathcal{C}([0,T); L^2(\mathbb{R}^N, |x|^{-\gamma}))$ is a solution of problem (2.2.6) if it is both subsolution and supersolution.

2.2.1 Fundamental inequalities.

In this subsection we collect the functional background needed to prove the Harnack inequality in the next subsection. As far as we know, these results are new in the framework of unbounded coefficients.

Recalling the spaces $Y^{s,\gamma}(\Omega)$ and $Y_0^{s,\gamma}(\Omega)$, we consider the operator

$$L_{\gamma,\Omega}(w)(x) := a_{N,s}\, \text{p.v.} \int_\Omega \frac{(w(x) - w(y))}{|x|^\gamma |y|^\gamma |x-y|^{N+2s}}\, dy,$$

and the associated scalar product

$$\langle L_{\gamma,\Omega}(w_1), w_2\rangle_{Y_0^{s,\gamma}(\Omega)} := \frac{a_{N,s}}{2}\iint_{\Omega\times\Omega}(w_1(x)-w_1(y))(w_2(x)-w_2(y))\,d\nu,$$

where w_1, $w_2 \in Y_0^{s,\gamma}(\Omega)$ and $d\nu$ was defined in (2.2.7). The case $\Omega = \mathbb{R}^N$ is allowed in these definitions. Indeed, $L_{\gamma,\mathbb{R}^N}(w)$ and $\langle\cdot,\cdot\rangle_{Y_0^{s,\gamma}(\mathbb{R}^N)}$ correspond to the operator L_γ defined in (2.2.4), and to (2.2.10) respectively.

Consider now the functions T_k and G_k defined in (1.1.8). By a straightforward adaptation, we can see that the properties collected in [134, Proposition 3] also hold in the weighted space $Y_0^{s,\gamma}$.

Proposition 2.2.4. *Let $\phi \in Y_0^{s,\gamma}(\Omega)$. Thus,*

- *If $\psi \in Lip(\mathbb{R})$, $\psi(0) = 0$, then $\psi(\phi) \in Y_0^{s,\gamma}(\Omega)$. In particular, for any $k \geqslant 0$, $T_k(\phi)$ and $G_k(\phi)$ belong to $Y_0^{s,\gamma}(\Omega)$.*
- *For any $k \geqslant 0$,*

$$\|T_k(\phi)\|_{Y_0^{s,\gamma}} \leqslant \langle L_{\gamma,\Omega}(\phi), T_k(v)\rangle_{Y_0^{s,\gamma}(\Omega)}, \qquad \|G_k(\phi)\|_{Y_0^{s,\gamma}} \leqslant \langle L_{\gamma,\Omega}(\phi), G_k(v)\rangle_{Y_0^{s,\gamma}(\Omega)}.$$

Likewise, in the sequel we will need two essential tools to work with these weighted spaces: an extension lemma and a weighted Sobolev embedding.

Lemma 2.2.5. *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain. Then for all $\phi \in Y^{s,\gamma}(\Omega)$, there exists $\tilde{\phi} \in Y^{s,\gamma}(\mathbb{R}^N)$ such that $\tilde{\phi}_{|\Omega} = \phi$ and*

$$\|\tilde{\phi}\|_{Y^{s,\gamma}(\mathbb{R}^N)} \leqslant C\|\phi\|_{Y^{s,\gamma}(\Omega)},$$

where $C = C(N, s, \Omega, \gamma) > 0$, and $\|\cdot\|_{Y^{s,\gamma}(\Omega)}$ was defined in (2.2.9).

The proof of this result follows using the same arguments of [11] (see also [82]). Furthermore, from [1] we know that the following Sobolev inequality holds.

Theorem 2.2.6. *Let $\phi \in \mathcal{C}_0^\infty(\Omega)$. Then, there exists a constant $C = C(N, s, \Omega, \gamma) > 0$ such that*

$$\Big(\int\limits_\Omega \frac{|\phi(x)|^{2_s^*}}{|x|^{\gamma 2_s^*}}\,dx\Big)^{\frac{2}{2_s^*}} \leqslant C(N, s, \Omega, \gamma)\iint_{\Omega\times\Omega}(\phi(x)-\phi(y))^2 d\nu. \tag{2.2.11}$$

Let us recall also that $Y_0^{s,\gamma}(\Omega)$ was defined as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{Y^{s,\gamma}(\Omega)}$. Thus, it is clear that if $\phi \equiv C \in Y_0^{s,\gamma}(\Omega)$, then $\phi \equiv 0$. In particular, if Ω is a bounded regular domain, we can prove the next Poincaré inequality.

Theorem 2.2.7. *There exists a positive constant $C = C(N, s, \Omega, \gamma)$ such that for any $\phi \in \mathcal{C}_0^\infty(\Omega)$ there holds*

$$C\int_\Omega \phi^2(x)d\mu \leqslant \iint_{\Omega\times\Omega}(\phi(x)-\phi(y))^2 d\nu.$$

Proof. If $\phi \equiv 0$, the inequality is indeed an identity, and it trivially follows. Thus, if we define

$$\lambda_1(\Omega) := \inf_{\{\phi\in\mathcal{C}_0^\infty(\Omega),\phi\not\equiv 0\}} \frac{\displaystyle\iint_{\Omega\times\Omega}(\phi(x)-\phi(y))^2 d\nu}{\displaystyle\int_\Omega \phi^2(x)d\mu},$$

to prove the lemma we need to check that $\lambda_1(\Omega) > 0$. We argue by contradiction, that is, let us suppose $\lambda_1(\Omega) = 0$. In such a case, there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{C}_0^\infty(\Omega)$ such that

$$\int_\Omega \phi_n^2(x)d\mu = 1 \text{ and } \iint_{\Omega\times\Omega}(\phi_n(x)-\phi_n(y))^2 d\nu \to 0 \text{ as } n\to\infty. \tag{2.2.12}$$

Thus, there exists $C > 0$, independent of n, such that $\|\phi_n\|_{Y^{s,\gamma}(\Omega)} \leqslant C$, and consequently one can ensure the existence of $\bar{\phi} \in Y^{s,\gamma}(\Omega)$ such that $\phi_n \rightharpoonup \bar{\phi}$ in $Y^{s,\gamma}(\Omega)$.

Define $\tilde{\phi}_n$ as the extension of ϕ_n given in Lemma 2.2.5. Then,

$$\|\tilde{\phi}_n\|_{Y^{s,\gamma}(\mathbb{R}^N)} \leqslant C(N,s,\Omega,\gamma)\|\phi_n\|_{Y^{s,\gamma}(\Omega)} \leqslant C,$$

and, since $\bar{\phi}_n$ coincides with ϕ in Ω, from Theorem 2.2.6 we obtain that

$$\int_\Omega \frac{|\phi_n|^{2_s^*}}{|x|^{2_s^*\gamma}}dx \leqslant \Big(\int_{\mathbb{R}^N}\frac{|\tilde{\phi}_n|^{2_s^*}}{|x|^{2_s^*\gamma}}dx\Big)^{\frac{1}{2_s^*}} \leqslant C\|\tilde{\phi}_n\|_{Y^{s,\gamma}(\mathbb{R}^N)} \leqslant C(N,s,\Omega,\gamma).$$

Using the fact that $Y^{s,\gamma}(\Omega) \subset Y^{s,0}(\Omega)$, it follows from [11] (see also [82]) that $\phi_n \to \bar{\phi}$ strongly in $L^2(\Omega)$. Hence, combining the estimates above and using Vitali's Lemma we obtain that, up to a subsequence,

$$\phi_n \to \bar{\phi} \text{ strongly in } L^2(\Omega,|x|^\gamma),$$

and thus,

$$\int_\Omega \bar{\phi}^2(x)d\mu = 1. \tag{2.2.13}$$

Moreover, by Fatou's Lemma, $\|\bar{\phi}\|_{Y^{s,\gamma}(\Omega)} \leqslant \|\phi_n\|_{Y^{s,\gamma}(\Omega)}$, and then from (2.2.12) it follows

$$\phi_n \to \bar{\phi} \text{ in } Y^{s,\gamma}(\Omega), \text{ and } \iint_{\Omega\times\Omega}(\bar{\phi}(x)-\bar{\phi}(y))^2 d\nu = 0,$$

Hence $\bar{\phi} \equiv C$, and since $\bar{\phi} \in Y_0^{s,\gamma}(\Omega)$, necessarily $\bar{\phi} \equiv 0$, a contradiction with (2.2.13). □

Moreover, we can prove an analogous result to Lemma 1.2.1 for the weighted operator.

Lemma 2.2.8. *If $w(x) := |x|^{-\theta}$, with $0 < \theta < \frac{N-2s-2\gamma}{2}$, then there exists a constant $C = C(N,s,\Omega,\gamma,\theta) > 0$ such that*

$$L_{\gamma,\mathbb{R}^N}(w)(x) = C\frac{w(x)}{|x|^{2s+2\gamma}} \quad a.e \ \ in\ \Omega.$$

Proof. By simplicity, we omit the constant $a_{N,s}$ and the principal value on the integral along this proof. We closely follow the arguments used by F. Ferrari and I.E. Verbitsky in [101].

Indeed, we set $r := |x|$ and $\rho := |y|$. Then $x = rx'$ and $y = \rho y'$, where $|x'| = |y'| = 1$, and therefore,

$$L_{\gamma,\mathbb{R}^N}(w)(x) \quad = \quad \frac{1}{|x|^\gamma}\int_0^{+\infty} \frac{(r^{-\theta}-\rho^{-\theta})\rho^{N-1}}{\rho^\gamma r^{N+2s}}\left(\int_{|y'|=1}\frac{dH^{N-1}(y')}{|x'-\frac{\rho}{r}y'|^{N+2s}}\right)d\rho.$$

Set now $\sigma := \dfrac{\rho}{r}$. Then,

$$L_{\gamma,\mathbb{R}^N}(w)(x) = \frac{w(x)}{|x|^{2s+2\gamma}}\int_0^{+\infty}(1-\sigma^{-\theta})\sigma^{N-\gamma-1}\left(\int_{|y'|=1}\frac{dH^{N-1}(y')}{|x'-\sigma y'|^{N+2s}}\right)d\sigma.$$

Define

$$K(\sigma) := \int_{|y'|=1}\frac{dH^{N-1}(y')}{|x'-\sigma y'|^{N+2s}},$$

then

$$K(\sigma) = 2\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}\int_0^\pi \frac{\sin^{N-2}(\eta)}{(1-2\sigma\cos(\eta)+\sigma^2)^{\frac{N+2s}{2}}}d\eta. \tag{2.2.14}$$

Thus

$$L_{\gamma,\mathbb{R}^N}(w) = \Lambda_{N,s,\gamma}\frac{w(x)}{|x|^{2s+2\gamma}},$$

where

$$\Lambda_{N,s,\gamma} = \int_0^{+\infty}(\sigma^\theta-1)\sigma^{N-\gamma-\theta-1}K(\sigma)d\sigma.$$

As in [101], taking into consideration the behavior of K near $\sigma = 1$ and at $+\infty$, we can prove that $|\Lambda_{N,s,\gamma}| < \infty$. To conclude we just have to show that $\Lambda_{N,s,\gamma} > 0$.

Since $K(\frac{1}{s}) = s^{N+2s}K(s)$ for all $s > 0$, we get

$$\begin{aligned}\Lambda_{N,s,\gamma} &= \int_0^1(\sigma^\theta-1)\sigma^{N-\gamma-\theta-1}K(\sigma)d\sigma + \int_1^\infty(\sigma^\theta-1)\sigma^{N-\gamma-\theta-1}K(\sigma)d\sigma\\ &= -\int_1^\infty(\xi^\theta-1)\xi^{2s+\gamma-1}K(\xi)d\xi + \int_1^\infty(\sigma^\theta-1)\sigma^{N-\gamma-\theta-1}K(\sigma)d\sigma\\ &= \int_1^\infty K(\sigma)(\sigma^\theta-1)(\sigma^{N-\gamma-\theta-1}-\sigma^{2s+\gamma-1})d\sigma.\end{aligned}$$

Hence, the result follows from the fact $0 < \theta < N-2s-2\gamma$. $\square$

Likewise, we can formulate a Picone type inequality for the weighted operator, in the spirit of Theorem 1.1.6.

Theorem 2.2.9. *Let $u, v \in Y_0^{s,\gamma}(\Omega)$, $u \gneq 0$, and assume that $L_{\gamma,\Omega}(u) =: \tilde{\nu}$ with $\tilde{\nu}$ in $L^1_{loc}(\mathbb{R}^N)$ and $\tilde{\nu} \gneqq 0$. Then it yields*

$$\langle L_{\gamma,\Omega}(u), \frac{v^2}{u}\rangle_{Y_0^{s,\gamma}(\Omega)} \leqslant \frac{a_{N,s}}{2}\|v\|^2_{Y_0^{s,\gamma}(\Omega)}. \tag{2.2.15}$$

Thanks to Proposition 2.2.4, this proof follows exactly in the same way as Theorem 1.1.6, only by replacing the space $H_0^s(\Omega)$ by $Y_0^{s,\gamma}$, since this was strongly based on a pointwise inequality, that also applies here.

As a consequence of this result, we are able to prove the next Hardy type inequality.

Theorem 2.2.10. *There exists a positive constant $C = C(N, s, \gamma)$ such that for every $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have*

$$C\int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^{2s+2\gamma}}dx \leqslant \iint_{\mathbb{R}^{2N}} (\phi(x)-\phi(y))^2 d\nu.$$

Proof. Let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ and define $w(x) := |x|^{-\theta}$, with $0 < \theta < \frac{N-2s-2\gamma}{2}$. Then, by Lemma 2.2.8,

$$L_{\gamma,\mathbb{R}^N}(w)(x) = C\frac{w(x)}{|x|^{2s+2\gamma}} \quad a.e \text{ in } \mathbb{R}^N.$$

Using (2.2.15) it follows that

$$\frac{a_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}\frac{dxdy}{|x|^\gamma|y|^\gamma} \geqslant \langle L_{\gamma,\mathbb{R}^N}(w), \frac{\phi^2}{w}\rangle_{Y_0^{s,\gamma}(\Omega)} = C(N,s,\gamma)\int_{\mathbb{R}^N}\frac{\phi^2(x)}{|x|^{2s+2\gamma}}dx.$$

Hence we conclude. □

In the case when Ω is a bounded domain, we can also obtain the corresponding inequality.

Theorem 2.2.11. *There exists a positive constant $C = C(N, s, \Omega, \gamma)$ such that*

$$C\int_{\Omega} \frac{\phi^2(x)}{|x|^{2s+2\gamma}}dx \leqslant \iint_{\Omega\times\Omega} (\phi(x)-\phi(y))^2 d\nu,$$

for every $\phi \in \mathcal{C}_0^\infty(\Omega)$.

Proof. Let $\phi \in \mathcal{C}_0^\infty(\Omega)$ and define $\tilde{\phi}$ to be the extension of ϕ to $\mathbb{R}^N$ given in Lemma 2.2.5. Then from Theorem 2.2.10, we get

$$\iint_{\mathbb{R}^{2N}} \frac{|\tilde{\phi}(x)-\tilde{\phi}(y)|^2}{|x-y|^{N+2s}}\frac{dxdy}{|x|^\gamma|y|^\gamma} \geqslant C(N,s,\gamma)\int_{\mathbb{R}^N}\frac{\tilde{\phi}^2(x)}{|x|^{2s+2\gamma}}dx.$$

Now, using the fact that $\tilde{\phi}_{|\Omega} = \phi$ and combining the results of Lemma 2.2.5 and Theorem 2.2.7, we reach the desired result. □

In the case of nonzero boundary conditions, we obtain the following version of the Hardy inequality.

Theorem 2.2.12. *There exists a positive constant $C(\Omega, N, s, \gamma)$ such that for every $\phi \in Y^{s,\gamma}(\Omega)$, we have*

$$C \int_\Omega \frac{\phi^2(x)}{|x|^{2s+2\gamma}} dx \leqslant \iint_{\Omega \times \Omega} (\phi(x) - \phi(y))^2 d\nu + \int_\Omega \phi^2(x) d\mu.$$

Proof. Fix $\phi \in Y^{s,\gamma}(\Omega)$ and define $\tilde{\phi}$ as the extension of ϕ to $\mathbb{R}^N$ given in Lemma 2.2.5. Then from Theorem 2.2.10, we get

$$\iint_{\mathbb{R}^{2N}} \frac{|\tilde{\phi}(x) - \tilde{\phi}(y)|^2}{|x-y|^{N+2s}} \frac{dxdy}{|x|^\gamma |y|^\gamma} \geqslant C(N, s, \gamma) \int_{\mathbb{R}^N} \frac{\tilde{\phi}^2(x)}{|x|^{2s+2\gamma}} dx \geqslant C(N, s, \gamma) \int_\Omega \frac{\phi^2(x)}{|x|^{2s+2\gamma}} dx.$$

Since $\|\tilde{\phi}\|_{Y^{s,\gamma}(\mathbb{R}^N)} \leqslant C(\Omega) \|\phi\|_{Y^{s,\gamma}(\Omega)}$, then the result follows. □

Moreover, in the particular case $\phi \in Y^{s,\gamma}(B_R)$, as an application of Theorem 2.2.12 we can prove the following improved inequality.

Theorem 2.2.13. *Let $R > 0$ and $\phi \in Y^{s,\gamma}(B_R)$. Then, there exists $C = C(N, s, R, \gamma) > 0$ such that*

$$C\Big(\int_{B_R} \frac{|\phi|^{2_s^*}}{|x|^{2_s^* \gamma}} dx \Big)^{\frac{2}{2_s^*}} \leqslant \iint_{B_R \times B_R} (\phi(x) - \phi(y))^2 d\nu + R^{-2s} \int_{B_R} \phi^2 d\mu. \tag{2.2.16}$$

Proof. We prove the result for $R = 1$, and then (2.2.16) follows by a scaling argument. We set $\phi_1(x) := \frac{\phi(x)}{|x|^\gamma}$. Then from [11] we know that

$$C\Big(\int_{B_1} |\phi_1|^{2_s^*} dx \Big)^{\frac{2}{2_s^*}} \leqslant \iint_{B_1 \times B_1} \frac{(\phi_1(x) - \phi_1(y))^2}{|x-y|^{N+2s}} dxdy + \int_{B_1} \phi_1^2 dx. \tag{2.2.17}$$

To get the desired result we just have to estimate the term

$$\iint_{B_1 \times B_1} \frac{(\phi_1(x) - \phi_1(y))^2}{|x-y|^{N+2s}} dxdy.$$

Since

$$(\phi_1(x) - \phi_1(y))^2 \quad = \quad \frac{(\phi(x) - \phi(y))^2}{|x|^\gamma |y|^\gamma} + \left(\frac{\phi^2(x)}{|x|^\gamma} - \frac{\phi^2(y)}{|y|^\gamma} \right) \left(\frac{1}{|x|^\gamma} - \frac{1}{|y|^\gamma} \right),$$

it follows that

$$\begin{aligned} &\iint_{B_1 \times B_1} \frac{(\phi_1(x) - \phi_1(y))^2}{|x-y|^{N+2s}} dxdy \\ &\qquad \leqslant \iint_{B_1 \times B_1} \frac{(\phi(x) - \phi(y))^2}{|x|^\gamma |y|^\gamma |x-y|^{N+2s}} dxdy + \int_{B_1} L_{0,B_1}(|x|^{-\gamma}) \frac{\phi^2}{|x|^\gamma} dx. \end{aligned}$$

Proceeding as in the proof of Lemma 2.2.8, we can prove that

$$L_{0,B_1}(|x|^{-\gamma}) \leqslant \frac{C}{|x|^{\gamma+2s}},$$

and hence

$$\iint_{B_1\times B_1}\frac{(\phi_1(x)-\phi_1(y))^2}{|x-y|^{N+2s}}dxdy \leqslant \iint_{B_1\times B_1}\frac{(\phi(x)-\phi(y))^2}{|x|^{\gamma}|y|^{\gamma}|x-y|^{N+2s}}dxdy + C\int_{B_1}\frac{\phi^2}{|x|^{2s+2\gamma}}dx.$$

Finally, using Theorem 2.2.12 and substituting $\phi(x)=|x|^{\gamma}\phi_1(x)$, we reach (2.2.16). □

We state now a weighted version of the Poincaré-Wirtinger inequality used in the proof of Lemma 2.2.22.

Theorem 2.2.14. *Let $w\in Y^{s,\gamma}(B_1)$ and assume that ψ is a radial decreasing function such that $\operatorname{supp}\psi\subset B_1$ and $0\lneqq\psi\leqslant 1$. Define*

$$W_{\psi}:=\frac{\int_{B_1}w(x)\psi(x)d\mu}{\int_{B_1}\psi(x)d\mu}.$$

Then, there exists $C:=C(N,s,\psi)>0$ such that

$$\int_{B_1}(w(x)-W_{\psi})^2\psi(x)d\mu\leqslant C\iint_{B_1\times B_1}(w(x)-w(y))^2\min\{\psi(x),\psi(y)\}d\nu.$$

Proof. Define $\Psi(x):=\dfrac{\psi(x)}{|x|^{2\gamma}}$, that is a radial decreasing function. Then using [86, Corollary 6] we get

$$\int_{B_1}(w(x)-\bar{W}_{\Psi})^2\Psi(x)dx\leqslant C\iint_{B_1\times B_1}\frac{(w(x)-w(y))^2}{|x-y|^{N+2s}}\min\{\Psi(x),\Psi(y)\}dxdy,$$

where

$$\bar{W}_{\Psi}=\frac{\int_{B_1}w(x)\Psi(x)dx}{\int_{B_1}\Psi(x)dx}.$$

Substituting Ψ by its value, we get

$$\int_{B_1}(w(x)-\bar{W}_{\Psi})^2\Psi(x)dx=\int_{B_1}(w(x)-W_{\psi})^2\psi(x)d\mu,$$

and

$$\begin{aligned}\iint_{B_1\times B_1}&\frac{(w(x)-w(y))^2}{|x-y|^{N+2s}}\min\{\Psi(x),\Psi(y)\}dxdy\\ &=\iint_{B_1\times B_1}\frac{(w(x)-w(y))^2}{|x-y|^{N+2s}}\min\left\{\frac{\psi(x)}{|x|^{2\gamma}},\frac{\psi(y)}{|y|^{2\gamma}}\right\}dxdy.\end{aligned}$$

Hence, to finish we just have to show that

$$\min\left\{\frac{\psi(x)}{|x|^{2\gamma}},\frac{\psi(y)}{|y|^{2\gamma}}\right\}\leqslant\frac{\min\{\psi(x),\psi(y)\}}{|x|^{\gamma}|y|^{\gamma}}\ \text{ in } B_1\times B_1.$$

Without loss of generality we can assume that $|x| \geqslant |y|$.

Define $H(s) := \frac{\psi(s)}{s^{2\gamma}}$, that is a decreasing function in $(0,1)$. Let $s_1 := |x|$ and $s_2 := |y|$, then

$$\min\Big\{\frac{\psi(x)}{|x|^{2\gamma}}, \frac{\psi(y)}{|y|^{2\gamma}}\Big\} = H(s_1).$$

Using that ψ is decreasing, we obtain that $\psi(s_1) \leqslant \psi(s_2)$. Thus

$$\frac{\min\{\psi(x), \psi(y)\}}{|x|^{\gamma}|y|^{\gamma}} = \frac{\psi(s_1)}{s_1^{\gamma} s_2^{\gamma}}.$$

Since $s_2 \leqslant s_1 \leqslant 1$, we conclude that $H(s_1) \leqslant \dfrac{\psi(s_1)}{s_1^{\gamma} s_2^{\gamma}}$ and the result follows. □

2.2.2 Weighted Harnack inequality.

The main result of this section is the next theorem.

Theorem 2.2.15. *(Weighted Weak Harnack inequality)*
Assume that f (resp. v_0) $\geqslant 0$ in $\Omega \times (0,T)$ (resp. in Ω). Let

$$v \in L^2(0,T; Y^{s,\gamma}(\mathbb{R}^N)) \cap \mathcal{C}([0,T); L^2(\mathbb{R}^N,\, |x|^{-\gamma}))$$

be a supersolution to (2.2.6) with $v \gneqq 0$ in $\mathbb{R}^N \times (0,T)$.

Then for any $q < 1 + \frac{2s}{N}$, we have

$$\Big(\int_{Q_1} v^q d\mu dt\Big)^{\frac{1}{q}} \leqslant C \inf_{Q_2} v, \tag{2.2.18}$$

where $Q_1 := B_r(x_0) \times (t_1, t_2), Q_2 := B_r(x_0) \times (t_3, t_4)$, with $0 < t_1 < t_2 < t_3 < t_4 < T$, $B_r(x_0) \subset \Omega$ and $C = C(N, r, t_1, t_2, t_3, t_4) > 0$.

The proof of this result follows the classical arguments by Moser (see [143]) with some necessary adaptations. In the context of parabolic fractional operators the precedent work is the paper by Felsinger and Kassmann, [96], where they prove this parabolic Harnack inequality for the fractional Laplacian, with the Lebesgue measure instead of our weighted measure (see Theorem 2.1.11). We will closely follow this work here, adapting the proofs to the weighted operator defined in (2.2.6).

First of all, we will need an iteration result, originally proved in [143] and extended by Bombieri and Giusti in [44] to the case of general measures in the elliptic setting (see also [151, Lemma 2.2.6]).

Lemma 2.2.16. *Let $\{U(r)\}_{\theta \leqslant r \leqslant 1}$ be a nondecreasing family of domains $U(r) \subset \mathbb{R}^{N+1}$, and let m, c_0 be positive constants, $\eta \in (0,1)$, $\theta \in [\frac{1}{2}, 1]$ and $0 < p_0 \leqslant +\infty$. Let w be a positive, measurable function defined on $U(1)$ satisfying*

$$\Big(\int_{U(r)} w^{p_0} d\mu\, dt\Big)^{\frac{1}{p_0}} \leqslant \Big(\frac{c_0}{(R-r)^m |U(1)|_{d\mu \times dt}}\Big)^{\frac{1}{p_0} - \frac{1}{p}} \Big(\int_{U(R)} w^p d\mu\, dt\Big)^{\frac{1}{p}},$$

for all $r, R \in [\theta, 1]$, $r < R$, *and for all* $p \in (0, \min\{\eta p_0, 1\})$.

Assume also that

$$\forall \rho > 0 : \; |U(1) \cap \{\log w > \rho\}|_{d\mu \times dt} \leqslant \frac{c_0 |U(1)|_{d\mu \times dt}}{\rho}.$$

Then there exists $C = C(\theta, \eta, c_0, m, p_0)$ *such that*

$$\Big(\int_{U(\theta)} w^{p_0} d\mu \, dt \Big)^{\frac{1}{p_0}} \leqslant C |U(1)|^{\frac{1}{p_0}}_{d\mu \times dt}.$$

Hereafter, we will make use of the following notation. Given $r > 0$, we define

$$I_-(r) := (-r^{2s}, 0), \qquad I_+(r) := (0, r^{2s}), \tag{2.2.19}$$

$$Q_-(r) := B_r(0) \times I_-(r), \qquad Q_+(r) := B_r(0) \times I_+(r). \tag{2.2.20}$$

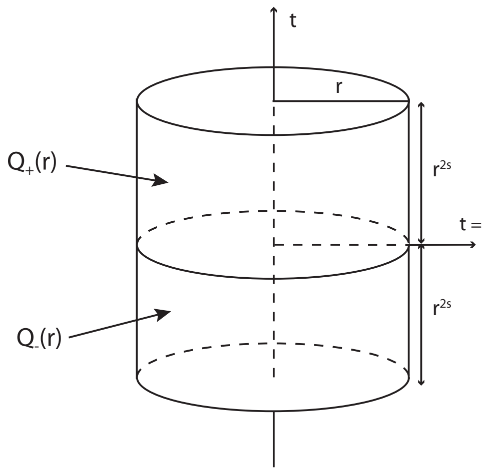


Figure 2.1: Cylindrical domains.

The first step to prove Theorem 2.2.15 is to establish the next estimate (see [96, Proposition 3.4]). Notice that we just have to consider the case where $B_r(x_0) = B_r(0)$. For simplicity, we will write B_r instead of $B_r(0)$.

Lemma 2.2.17. *Assume that* $\frac{1}{2} \leqslant r < R \leqslant 1$ *and let* $p > 0$. *Consider* $v \gneqq 0$, *a supersolution to* (2.2.6), *then*

$$\Big(\iint_{Q_-(r)} v^{-\tau p} d\mu dt \Big)^{\frac{1}{\tau}} \leqslant A \iint_{Q_-(r)} v^{-p} d\mu dt \tag{2.2.21}$$

where $\tau := 1 + \frac{2s}{N}$, *and*

$$A := A(N, s, p, r, R, \gamma) = C(N, s, \gamma)(p+1)^2 \left(\frac{1}{(R-r)^{2s}} + \frac{1}{R^{2s} - r^{2s}} \right).$$

Proof. Without loss of generality we can assume that $v \geqslant \varepsilon > 0$ in $Q_-(R)$ (otherwise we can deal with $v+\varepsilon$ and let $\varepsilon \to 0$ at the end). Let $q > 1$, $R > r$ and $\psi : \mathbb{R}^N \to [0,1]$ defined by

$$\psi(x) := \max\left\{\min\left\{\frac{R-|x|}{R-r}, 1\right\}, 0\right\}. \tag{2.2.22}$$

Notice that $\psi \in Y_0^{s,\gamma}(B_R) \cap L^\infty(B_R)$, $\operatorname{supp}(\psi) \subseteq B_R$ with $r < R$, $\psi = 1$ in B_r and

$$\frac{(\psi(x)-\psi(y))^2}{|x-y|^2} \leqslant \frac{C}{(R-r)^2}. \tag{2.2.23}$$

Using $\psi^{q+1}v^{-q}$ as a test function in (2.2.6), it follows that

$$\int_{B_R} \psi^{q+1}v^{-q}v_t d\mu + \frac{a_{N,s}}{2}\iint_{\mathbb{R}^{2N}} (v(x,t)-v(y,t))(\psi^{q+1}(x)v^{-q}(x,t)-\psi^{q+1}(y)v^{-q}(y,t))d\nu \geqslant 0.$$

Hence, setting $a := v(y,t)$, $b := v(x,t)$, $\tau_1 := \psi(y)$ and $\tau_2 := \psi(x)$ in [96, Lemma 3.3], it can be deduced that

$$\begin{aligned}
&\frac{1}{q-1}\int_{B_R} \psi^{q+1}(v^{1-q})_t d\mu \\
&\quad + \frac{1}{q-1}\frac{a_{N,s}}{2}\iint_{B_R\times B_R} \psi(x)\psi(y)\left(\left(\frac{v(x,t)}{\psi(x)}\right)^{\frac{1-q}{2}} - \left(\frac{v(y,t)}{\psi(y)}\right)^{\frac{1-q}{2}}\right)^2 d\nu \\
&\qquad \leqslant \vartheta(q)\frac{a_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \left(\left(\frac{v(x,t)}{\psi(x)}\right)^{1-q} + \left(\frac{v(y,t)}{\psi(y)}\right)^{1-q}\right)(\psi(x)-\psi(y))^2 d\nu,
\end{aligned}$$

where $\vartheta(q) \leqslant Cq$. Using the positivity of ψ and the fact that $\psi \equiv 1$ in B_r, we obtain

$$\begin{aligned}
&\int_{B_R} \psi^{q+1}(v^{1-q})_t d\mu + \frac{a_{N,s}}{2}\iint_{B_r\times B_r} (v^{\frac{1-q}{2}}(x,t) - v^{\frac{1-q}{2}}(y,t))^2 d\nu \\
&\quad \leqslant Cq^2\frac{a_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \left(\left(\frac{v(x,t)}{\psi(x)}\right)^{1-q} + \left(\frac{v(y,t)}{\psi(y)}\right)^{1-q}\right)(\psi(x)-\psi(y))^2 d\nu.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \left(\left(\frac{v(x,t)}{\psi(x)}\right)^{1-q} + \left(\frac{v(y,t)}{\psi(y)}\right)^{1-q}\right)(\psi(x)-\psi(y))^2 d\nu \\
&\qquad \leqslant 2\int_{B_R}\int_{B_R} \frac{v^{1-q}(x,t)}{|x|^\gamma}\frac{(\psi(x)-\psi(y))^2 dxdy}{|y|^\gamma|x-y|^{N+2s}} \\
&\qquad\quad + 2\int_{\mathbb{R}^N\setminus B_R}\int_{B_R} \frac{v^{1-q}(x,t)}{|x|^\gamma}\frac{(\psi(x)-\psi(y))^2 dxdy}{|y|^\gamma|x-y|^{N+2s}}.
\end{aligned}$$

We set

$$I := \int_{B_R}\int_{B_R} \frac{v^{1-q}(x,t)}{|x|^\gamma}\frac{(\psi(x)-\psi(y))^2 dxdy}{|y|^\gamma|x-y|^{N+2s}},$$

and

$$J := \int_{\mathbb{R}^N\setminus B_R}\int_{B_R} \frac{v^{1-q}(x,t)}{|x|^\gamma}\frac{(\psi(x)-\psi(y))^2dxdy}{|y|^\gamma|x-y|^{N+2s}}.$$

Let begin by estimating J. Considering that $|y|^{-\gamma}\leqslant |x|^{-\gamma}$ for $x\in B_R$ and $y\in\mathbb{R}^N\setminus B_R$, and using Fubini, we reach that

$$\begin{aligned}
J &\leqslant \int_{B_R}\frac{v^{1-q}(x,t)}{|x|^{2\gamma}}\int_{\{\mathbb{R}^N\setminus B_R\}\cap\{|x-y|>R-r\}}\frac{(\psi(x)-\psi(y))^2dydx}{|x-y|^{N+2s}}\\
&\quad+\int_{B_R}\frac{v^{1-q}(x,t)}{|x|^{2\gamma}}\int_{\{\mathbb{R}^N\setminus B_R\}\cap\{|x-y|\leqslant R-r\}}\frac{(\psi(x)-\psi(y))^2dydx}{|x-y|^{N+2s}}\\
&=: J_1+J_2.
\end{aligned}$$

Since $0\leqslant\psi(x)\leqslant 1$, we get

$$\begin{aligned}
J_1 &\leqslant 4\int_{B_R}\frac{v^{1-q}(x,t)}{|x|^{2\gamma}}\int_{\{\mathbb{R}^N\setminus B_R\}\cap\{|x-y|>R-r\}}\frac{dydx}{|x-y|^{N+2s}}\\
&\leqslant 4\int_{B_R}\frac{v^{1-q}(x,t)}{|x|^{2\gamma}}dx\int_{R-r}^{\infty}\rho^{-1-2s}d\rho\\
&\leqslant \frac{C}{(R-r)^{2s}}\int_{B_R}\frac{v^{1-q}(x,t)}{|x|^{2\gamma}}dx.
\end{aligned}$$

We estimate now the term J_2. Using (2.2.23), it follows that

$$\begin{aligned}
J_2 &\leqslant \frac{1}{(R-r)^2}\int_{B_R}\frac{v^{1-q}(x,t)}{|x|^{2\gamma}}\int_{\{\mathbb{R}^N\setminus B_R\}\cap\{|x-y|\leqslant R-r\}}\frac{|x-y|^2dydx}{|x-y|^{N+2s}}\\
&\leqslant \frac{1}{(R-r)^2}\int_{B_R}\frac{v^{1-q}(x,t)}{|x|^{2\gamma}}dx\int_0^{R-r}\rho^{1-2s}d\rho\\
&\leqslant \frac{C}{(R-r)^{2s}}\int_{B_R}\frac{v^{1-q}(x,t)}{|x|^{2\gamma}}dx.
\end{aligned}$$

Hence

$$J\leqslant\frac{C}{(R-r)^{2s}}\int_{B_R}v^{1-q}(x,t)d\mu.$$

We deal now with I. Using the definition of ψ, we get easily that

$$\begin{aligned}
I &= \int\int_{B_R\times B_R\setminus B_r\times B_r}\frac{v^{1-q}(x,t)}{|x|^\gamma}\frac{(\psi(x)-\psi(y))^2dxdy}{|y|^\gamma|x-y|^{N+2s}}\\
&= \int_{B_R\setminus B_r}\int_{B_r}\frac{v^{1-q}(x,t)}{|x|^\gamma}\frac{(\psi(x)-\psi(y))^2dxdy}{|y|^\gamma|x-y|^{N+2s}}\\
&\quad+\int_{B_R\setminus B_r}\int_{B_R\setminus B_r}\frac{v^{1-q}(x,t)}{|x|^\gamma}\frac{(\psi(x)-\psi(y))^2dxdy}{|y|^\gamma|x-y|^{N+2s}}\\
&\quad+\int_{B_r}\int_{B_R\setminus B_r}\frac{v^{1-q}(x,t)}{|x|^\gamma}\frac{(\psi(x)-\psi(y))^2dxdy}{|y|^\gamma|x-y|^{N+2s}}\\
&=: I_1+I_2+I_3.
\end{aligned}$$

Let begin by estimating I_1. Since $(x,y) \in B_r \times B_R \backslash B_r$, then $|x| \leqslant |y|$, hence

$$\begin{aligned} I_1 &= \int_{B_r} \int_{B_R \backslash B_r} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} \frac{(\psi(x)-\psi(y))^2 dydx}{|x-y|^{N+2s}} \\ &\leqslant \int_{B_r} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} \int_{\{B_R \backslash B_r\} \cap \{|x-y|>R-r\}} \frac{(\psi(x)-\psi(y))^2 dydx}{|x-y|^{N+2s}} \\ &+ \int_{B_r} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} \int_{\{B_R \backslash B_r\} \cap \{|x-y|\leqslant R-r\}} \frac{(\psi(x)-\psi(y))^2 dydx}{|x-y|^{N+2s}}, \end{aligned}$$

and, repeating the computations for J_1 and J_2, we conclude that

$$I_1 \leqslant \frac{C}{(R-r)^{2s}} \int_{B_R} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} dx.$$

We deal now with I_2. Since $\frac{1}{2} \leqslant r \leqslant |x|, |y| \leqslant R < 1$, then $|x|^{-\gamma} < |x|^{-2\gamma}$ and $|y|^{-\gamma} \leqslant 2^\gamma$. Therefore,

$$\begin{aligned} I_2 &\leqslant 2^\gamma \int_{B_R \backslash B_r} \int_{B_R \backslash B_r} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} \frac{(\psi(x)-\psi(y))^2 dxdy}{|x-y|^{N+2s}} \\ &\leqslant \int_{B_R \backslash B_r} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} \int_{\{B_R \backslash B_r\} \cap \{|x-y|>R-r\}} \frac{(\psi(x)-\psi(y))^2 dydx}{|x-y|^{N+2s}} \\ &+ \int_{B_R \backslash B_r} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} \int_{\{B_R \backslash B_r\} \cap \{|x-y|\leqslant R-r\}} \frac{(\psi(x)-\psi(y))^2 dydx}{|x-y|^{N+2s}}, \end{aligned}$$

and again, from here we obtain that

$$I_2 \leqslant \frac{C}{(R-r)^{2s}} \int_{B_R} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} dx.$$

Finally, let us consider the term I_3.

$$\begin{aligned} I_3 &= \int_{r \leqslant |x| \leqslant R} \frac{v^{1-q}(x,t)}{|x|^\gamma} \left(\int_{|y| \leqslant \frac{|x|}{2}} \frac{(\psi(x)-\psi(y))^2}{|x-y|^{N+2s}|y|^\gamma} dy \right) dx \\ &+ \int_{r \leqslant |x| \leqslant R} \frac{v^{1-q}(x,t)}{|x|^\gamma} \left(\int_{\frac{|x|}{2} \leqslant |y| \leqslant r} \frac{(\psi(x)-\psi(y))^2}{|x-y|^{N+2s}|y|^\gamma} dy \right) dx \\ &= I_{31} + I_{32}. \end{aligned}$$

If $|y| \leqslant \frac{|x|}{2}$, then $|x-y| \geqslant \frac{|x|}{2} \geqslant \frac{r}{2} \geqslant \frac{1}{4}$, and thus,

$$\begin{aligned} I_{31} &\leqslant C \int_{r \leqslant |x| \leqslant R} \frac{v^{1-q}(x,t)}{|x|^\gamma} \left(\int_{|y| \leqslant \frac{|x|}{2}} \frac{1}{|y|^\gamma} dy \right) dx \\ &\leqslant C \int_{r \leqslant |x| \leqslant R} \frac{v^{1-q}(x,t)}{|x|^\gamma} \left(\int_0^{\frac{|x|}{2}} \rho^{N-1-\gamma} d\rho \right) dx \\ &\leqslant \frac{C}{(R-r)^{2s}} \int_{B_R} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} dx. \end{aligned}$$

To estimate I_{32}, we use the fact that $|y|^{-\gamma} \leqslant 2^{\gamma}|x|^{-\gamma}$. Hence,

$$\begin{aligned}
I_{32} &\leqslant C \int_{r\leqslant|x|\leqslant R} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} \left(\int_{\frac{|x|}{2}\leqslant|y|\leqslant r} \frac{(\psi(x)-\psi(y))^2}{|x-y|^{N+2s}} dy \right) dx \\
&\leqslant C \int_{r\leqslant|x|\leqslant R} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} \left(\int_{\{\frac{|x|}{2}\leqslant|y|\leqslant r\}\cap\{|x-y|>R-r\}} \frac{(\psi(x)-\psi(y))^2}{|x-y|^{N+2s}} dy \right) dx \\
&\quad + C \int_{r\leqslant|x|\leqslant R} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} \left(\int_{\{\frac{|x|}{2}\leqslant|y|\leqslant r\}\cap\{|x-y|\leqslant R-r\}} \frac{(\psi(x)-\psi(y))^2}{|x-y|^{N+2s}} dy \right) dx,
\end{aligned}$$

and we conclude, as in the previous cases, that

$$I_{32} \leqslant \frac{C}{(R-r)^{2s}} \int_{B_R} \frac{v^{1-q}(x,t)}{|x|^{2\gamma}} dx.$$

Combining the estimates above, from (2.2.2) we deduce

$$\begin{aligned}
\int_{B_R} \psi^{q+1}(v^{1-q})_t d\mu &+ \frac{a_{N,s}}{2} \int_{B_r}\int_{B_r} (v^{\frac{1-q}{2}}(x,t) - v^{\frac{1-q}{2}}(y,t))^2 d\nu \\
&\leqslant \frac{Cq^2}{(R-r)^{2s}} \frac{a_{N,s}}{2} \int_{B_R} v^{1-q} d\mu,
\end{aligned}$$

with $C = C(q,N,s,\gamma,R,r)$.

Set now

$$\theta(t) := \max\left\{ \min\left\{ \frac{t+R^{2s}}{R^{2s}-r^{2s}}, 1 \right\}, 0 \right\}.$$

Multiplying the last inequality by θ,

$$\begin{aligned}
\int_{B_R} \psi^{q+1}(v^{1-q}(x,t)\theta(t))_t d\mu &+ \frac{a_{N,s}}{2} \iint_{B_r\times B_r} (v^{\frac{1-q}{2}}(x,t) - v^{\frac{1-q}{2}}(y,t))^2 d\nu \\
&\leqslant \frac{Cq^2}{(R-r)^{2s}} \frac{a_{N,s}}{2} \int_{B_R} v^{1-q}(x,t) d\mu + \int_{B_R} \psi^{q+1} v^{1-q}(x,t)|\theta'(t)| d\mu,
\end{aligned}$$

and integrating in time in $(-R^{2s}, t)$ with $t \in (-r^{2s}, 0)$, we get

$$\begin{aligned}
\int_{B_R} \psi^{q+1} v^{1-q}(x,t)\theta(t) d\mu &+ \frac{a_{N,s}}{2} \int_{-R^{2s}}^{t} \iint_{B_r\times B_r} (v^{\frac{1-q}{2}}(x,\sigma) - v^{\frac{1-q}{2}}(y,\sigma))^2 d\nu\, d\sigma \\
&\leqslant \frac{Cq^2}{(R-r)^{2s}} \frac{a_{N,s}}{2} \int_{B_R} v^{1-q}(x,\sigma) d\mu\, d\sigma + \int_{B_R} \psi^{q+1} v^{1-q}(x,\sigma)|\theta'(\sigma)| d\mu\, d\sigma.
\end{aligned}$$

Since this inequality holds for every $t \in (-r^{2s}, 0)$, and noticing that

$$\theta(\sigma) = 1 \text{ for } \sigma \geqslant -r^{2s} \text{ and } |\theta'(t)| \leqslant \frac{1}{R^{2s}-r^{2s}},$$

it follows that

$$\begin{aligned}\sup_{t\in I_-(r)}\int_{B_r} v^{1-q}d\mu + \iint_{Q_-(r)}\int_{B_r}(v^{\frac{1-q}{2}}(x,\sigma)-v^{\frac{1-q}{2}}(y,\sigma))^2 d\nu d\sigma \\ \leqslant C(N,s,\gamma)q^2\left(\frac{1}{(R-r)^{2s}}+\frac{1}{R^{2s}-r^{2s}}\right)\iint_{Q_-(r)} v^{1-q}d\mu\, d\sigma.\end{aligned} \tag{2.2.24}$$

Recalling that $\tau := 1+\frac{2s}{N}$, and defining $w := v^{\frac{1-q}{2}}$, by Hölder's inequality we get

$$\begin{aligned}\iint_{Q_-(r)} w^{2\tau}d\mu dt &= \iint_{Q_-(r)} w^2 w^{\frac{4s}{N}}d\mu dt \\ &\leqslant \int_{I_-(r)}\left(\int_{B_r} w^2 d\mu\right)^{\frac{2s}{N}}\left(\int_{B_r} w^{2_s^*}d\mu\right)^{\frac{2}{2_s^*}} dt.\end{aligned}$$

Since $\gamma > 0$ and $R \leqslant 1$, we conclude that

$$\iint_{Q_-(r)} w^{2\tau}d\mu dt \leqslant \int_{I_-(r)}\left(\int_{B_r} w^2 d\mu\right)^{\frac{2s}{N}}\left(\int_{B_r}\frac{w^{2_s^*}}{|x|^{2_s^*\gamma}}dx\right)^{\frac{2}{2_s^*}} dt.$$

Now, using the Sobolev inequality obtained in Theorem 2.2.13,

$$\begin{aligned}\iint_{Q_-(r)} w^{2\tau}d\mu dt &\leqslant C\sup_{t\in I_-(r)}\left(\int_{B_r} w^2(x,t)d\mu\right)^{\frac{2s}{N}} \\ &\times\left(\iint_{Q_-(r)}(w(x,t)-w(y,t))^2 d\nu dt + r^{-2s}\iint_{Q_-(r)} w^2 d\mu dt\right).\end{aligned}$$

Applying (2.2.24) twice at this inequality, and recalling that $\frac{1}{2}\leqslant r\leqslant 1$, it can be checked that

$$\iint_{Q_-(r)} w^{2\tau}d\mu dt \leqslant A(q,r,R,N,s,\gamma)\left(\iint_{Q_-(r)} w^2 d\mu dt\right)^{\tau}$$

where

$$A(q,r,R,N,s,\gamma) = C(N,s,\gamma)q^2\left(\frac{1}{(R-r)^{2s}}+\frac{1}{R^{2s}-r^{2s}}\right)^{\tau}.$$

Setting $p := q-1$, we conclude the proof. $\square$

As an application of the previous estimate, we reach a control of $\sup_{Q_-(r)} v^{-1}$. More precisely, we have the following result.

Lemma 2.2.18. *Assume that $\frac{1}{2}\leqslant r < R \leqslant 1$ and $p\in(0,1]$. Then, there exists a constant $C = C(N,s,\gamma) > 0$ such that every $v \gneq 0$ supersolution to the problem (2.2.6) satisfies*

$$\sup_{Q_-(r)} v^{-1} \leqslant \left(\frac{C}{\bar{A}(r,R)}\right)^{\frac{1}{p}}\left(\int_{Q_-(R)} v^{-p}d\mu dt\right)^{\frac{1}{p}}, \tag{2.2.25}$$

where

$$\bar{A}(r,R) = \begin{cases} (R-r)^{N+2s} & \text{if} \quad s \geqslant \frac{1}{2}, \\ (R^{2s}-r^{2s})^{\frac{N+2s}{2s}} & \text{if} \quad s < \frac{1}{2}. \end{cases}$$

Proof. Let us consider

$$\mathcal{M}(r,p) := \left(\iint_{Q_-(r)} v^{-p} d\mu \, dt \right)^{\frac{1}{p}}.$$

By Lemma 2.2.17, we have

$$\mathcal{M}(r,\tau p) \leqslant A^{\frac{1}{p}} \mathcal{M}(R,p),$$

where $\tau := 1 + \frac{2s}{N}$. We construct now two sequences $\{r_i\}_{i\in\mathbb{N}}$ and $\{p_i\}_{i\in\mathbb{N}}$ by setting

$$\begin{aligned} r_0 &:= R > r_1 > r_2 > ... > r, \\ p_m &:= p\tau^m, \quad m \in \mathbb{N}. \end{aligned}$$

Thus, iterating Lemma 2.2.17, we obtain

$$\mathcal{M}(r,p_{m+1}) \leqslant \mathcal{M}(r_{m+1},p_{m+1}) \leqslant A_m^{\frac{1}{\tau^m p}} \mathcal{M}(r_m,p_m) \leqslant \mathcal{M}(r_0,p) \left(\prod_{j=0}^{m} A_j^{1/\tau^j} \right)^{1/p}$$

with $A_j := C(p_j+1)^2 \left((r_j - r_{j+1})^{-2s} + (r_j^{2s} - r_{j+1}^{2s})^{-1} \right)$. Letting $m \to \infty$ it yields,

$$\left(\sup_{Q_-(r)} v^{-1} \right)^p = \sup_{Q_-(r)} v^{-p} = \lim_{m\to\infty} \mathcal{M}^p(r,p_m) \leqslant \mathcal{M}^p(R,p) \prod_{j=0}^{\infty} A_j^{1/\tau^j}.$$

Following exactly the arguments at the end of the proof of [96, Theorem 3.5], we conclude that

$$\prod_{j=0}^{\infty} A_j^{1/\tau^j} \leqslant \frac{C}{(R-r)^{N+2s} + (R^{2s}-r^{2s})^{\frac{N+2s}{2s}}},$$

for some $C = C(N,s,\gamma)$, and the result follows. $\square$

We prove now a control for small positive exponents (see [96, Proposition 3.6] for a detailed proof with the Lebesgue measure).

Lemma 2.2.19. *Suppose that* $\frac{1}{2} \leqslant r < R \leqslant 1$, *and fix* $q \in (0, \tau^{-1}]$, *with* $\tau := 1 + \frac{2s}{N}$. *Then, if* $v \gneq 0$ *is a supersolution to* (2.2.6), *we have*

$$\left(\iint_{Q_+(r)} v^{q\tau} d\mu dt \right)^{\frac{1}{\tau}} \leqslant \alpha \iint_{Q_+(r)} v^q d\mu dt, \tag{2.2.26}$$

where

$$\alpha = \alpha(N,s,\gamma,r,R) = C(N,s,\gamma) \left(\frac{1}{(R-r)^{2s}} + \frac{1}{R^{2s}-r^{2s}} \right).$$

Proof. As in the proof of Lemma 2.2.17, we start assuming that $v \geqslant \varepsilon > 0$ in $Q_+(R)$.

Set $a := 1-q \in [1-\tau^{-1}, 1)$ and consider $v^{-a}\psi^2$ as a test function in (2.2.6), with ψ defined in (2.2.22). Thus,

(2.2.27)
$$-\int_{B_R} \psi^2 v^{-a} v_t \, d\mu + \frac{a_{N,s}}{2} \iint_{\mathbb{R}^{2N}} (v(x,t)-v(y,t))(\psi^2(y)v^{-a}(y,t) - \psi^2(x)v^{-a}(x,t)) \, d\nu \leqslant 0.$$

Moreover, it can be checked that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} (v(x,t)-v(y,t))(\psi^2(y)v^{-a}(y,t) - \psi^2(x)v^{-a}(x,t)) \, d\nu \\
&\qquad = \int_{B_R}\int_{B_R} (v(x,t)-v(y,t))(\psi^2(y)v^{-a}(y,t) - \psi^2(x)v^{-a}(x,t)) \, d\nu \\
&\qquad\quad + 2\int_{\mathbb{R}^N\setminus B_R}\int_{B_R} (v(x,t)-v(y,t))(-\psi^2(x)v^{-a}(x,t)) \, d\nu,
\end{aligned}$$

and using the positivity of v and ψ, that $|x| < |y|$ for $x \in B_R$, $y \in \mathbb{R}^N \setminus B_R$, and that ψ vanishes in $\mathbb{R}^N \setminus B_R$, there holds

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_R}\int_{B_R} (v(x,t)-v(y,t))(-\psi^2(x)v^{-a}(x,t)) \, d\nu \\
&\qquad \geqslant -\int_{\mathbb{R}^N\setminus B_R}\int_{B_R} v^{1-a}(x,t)(\psi(x)-\psi(y))^2 \, d\nu \\
&\qquad \geqslant -\int_{B_R} \frac{v^{1-a}(x,t)}{|x|^{2\gamma}} \int_{\{|x-y|>R-r\}} \frac{(\psi(x)-\psi(y))^2 \, dydx}{|x-y|^{N+2s}} \\
&\qquad\quad -\int_{B_R} \frac{v^{1-a}(x,t)}{|x|^{2\gamma}} \int_{\{|x-y|\leqslant R-r\}} \frac{(\psi(x)-\psi(y))^2 \, dydx}{|x-y|^{N+2s}}.
\end{aligned}$$

Proceeding as in the proof of Lemma 2.2.17, i.e., using the boundedness of ψ in the first integral and (2.2.23) in the second one, it follows

$$\int_{\mathbb{R}^N\setminus B_R}\int_{B_R} (v(x,t)-v(y,t))(-\psi^2(x)v^{-a}(x,t)) \, d\nu \geqslant -\frac{C}{(R-r)^{2s}} \int_{B_R} v^{1-a} \, d\mu.$$

Thus, noticing that $v^{-a}v_t = \frac{1}{1-a}(v^{1-a})_t$, from (2.2.27) we deduce

$$\begin{aligned}
&\frac{-1}{1-a}\int_{B_R} \psi^2 (v^{1-a})_t \, d\mu \\
&\qquad + \frac{a_{N,s}}{2} \iint_{B_R\times B_R} (v(x,t)-v(y,t))(\psi^2(y)v^{-a}(y,t) - \psi^2(x)v^{-a}(x,t)) \, d\nu \\
&\qquad \leqslant \frac{C}{(R-r)^{2s}} \int_{B_R} v^{1-a} \, d\mu.
\end{aligned}$$

Applying again the pointwise inequalities in [96, Lemma 3.3] we obtain

$$\frac{-1}{1-a}\int_{B_R}\psi^2(v^{1-a})_t\,d\mu+\vartheta_1(a)\frac{a_{N,s}}{2}\iint_{B_R\times B_R}(\psi(x)v^{\frac{1-a}{2}}(x,t)-\psi(y)v^{\frac{1-a}{2}}(y,t))^2\,d\nu$$
$$\leqslant\frac{C}{(R-r)^{2s}}\int_{B_R}v^{1-a}\,d\mu+\vartheta_2(a)\frac{a_{N,s}}{2}\iint_{B_R\times B_R}(\psi(x)-\psi(y))^2(v^{1-a}(x,t)+v^{1-a}(y,t))\,d\nu.$$

On the other hand,

$$\begin{aligned}\iint_{B_R\times B_R}&(\psi(x)-\psi(y))^2(v^{1-a}(x,t)+v^{1-a}(y,t))\,d\nu\\&=2\int_{B_R}\frac{v^{1-a}(x,t)}{|x|^\gamma}\left(\int_{|y|\leqslant\frac{|x|}{2}}\frac{(\psi(x)-\psi(y))^2}{|x-y|^{N+2s}|y|^\gamma}\,dy\right)dx\\&\quad+2\int_{B_R}\frac{v^{1-a}(x,t)}{|x|^\gamma}\left(\int_{\frac{|x|}{2}\leqslant|y|\leqslant R}\frac{(\psi(x)-\psi(y))^2}{|x-y|^{N+2s}|y|^\gamma}\,dy\right)dx\\&\leqslant\frac{C}{(R-r)^{2s}}\int_{B_R}v^{1-a}(x,t)\,d\mu,\end{aligned}$$

where the last inequality is obtained as a consequence of the properties of ψ, reproducing the arguments used in the proof of Lemma 2.2.17 to bound I_3. Therefore,

(2.2.28)
$$\begin{aligned}-\int_{B_R}\psi^2(v^{1-a})_t\,d\mu+(1-a)\vartheta_1(a)\frac{a_{N,s}}{2}\iint_{B_R\times B_R}(\psi(x)v^{\frac{1-a}{2}}(x,t)-\psi(y)v^{\frac{1-a}{2}}(y,t))^2\,d\nu\\\leqslant\frac{C(N,s,\gamma)}{(R-r)^{2s}}(1-a)(1+\vartheta_2(a))\int_{B_R}v^{1-a}\,d\mu.\end{aligned}$$

Define now the function

$$\chi(t):=\max\left\{\min\left\{\frac{R^{2s}-t}{R^{2s}-r^{2s}},1\right\},0\right\}.$$

Analogously to Lemma 2.2.17, we multiply by χ in (2.2.28), and we integrate in time between $t\in I_+(r)$ and R^{2s} to obtain

$$\begin{aligned}\sup_{t\in I_+(r)}&\int_{B_R}v^{1-a}\,d\mu\\&+(1-a)\vartheta_1(a)\iint_{Q_+(r)}\int_{B_R}(\psi(x)v^{\frac{1-a}{2}}(x,\sigma)-\psi(y)v^{\frac{1-a}{2}}(y,\sigma))^2\,d\nu d\sigma\\&\leqslant C(N,s,\gamma)\left((1-a)+\frac{(1-a)\vartheta_2(a)}{(R-r)^{2s}}+\frac{1}{(R^{2s}-r^{2s})}\right)\iint_{Q_+(r)}v^{1-a}\,d\mu d\sigma,\end{aligned}$$

and again, the result follows as a consequence of Theorem 2.2.13 (see [96, Lemma 3.3] for the details concerning to the precise value of α in (2.2.26)). □

In the same way as we obtained Lemma 2.2.18 by iterating Lemma 2.2.17, thanks to the previous result we can bound from above the L^1 norm of a supersolution by certain L^p norms. More precisely,

Lemma 2.2.20. *Assume that* $\frac{1}{2} \leqslant r < R \leqslant 1$ *and* $q \in (0, \tau^{-1})$, *with* $\tau := 1 + \frac{2s}{N}$. *Then, every supersolution* $v \gneq 0$ *of problem* (2.2.6) *satisfies*

$$\iint_{Q_+(r)} v \, d\mu dt \leqslant \left(\frac{C}{|Q_+(1)|_{d\mu \times dt} \overline{\alpha}(r,R)} \right)^{\frac{1-q}{q}} \left(\iint_{Q_+(r)} v^q d\mu dt \right)^{\frac{1}{q}}, \tag{2.2.29}$$

where $C = C(N, s, \gamma) > 0$ *and*

$$\overline{\alpha}(r,R) = \begin{cases} (R-r)^{\omega_1} & \text{if } s \geqslant \frac{1}{2}, \\ (R^{2s} - r^{2s})^{\omega_2} & \text{if } s < \frac{1}{2}, \end{cases}$$

with $\omega_1, \omega_2 > 0$ *depending only on* s, N.

Proof. Define

$$\mathcal{H}(r,q) = \Big(\iint_{Q_+(r)} v^q d\mu dt \Big)^{\frac{1}{q}}.$$

Thus, from (2.2.26) we can write

$$\mathcal{H}(r, \tau q) \leqslant \alpha^{\frac{1}{q}} \mathcal{H}(R, q).$$

Let consider first the case $s \geqslant \frac{1}{2}$, and define

$$q_j := \tau^{-j}, \qquad r_j := r + \frac{R-r}{2^j}. \tag{2.2.30}$$

For such a range of s, there holds

$$\frac{1}{(r_{n-1} - r_n)^{2s}} + \frac{1}{(r_{n-1}^{2s} - r_n^{2s})} \leqslant \frac{2}{(r_{n-1} - r_n)^{2s}},$$

and thus, by Lemma 2.2.19 and (2.2.30), it can be easily checked that

$$\mathcal{H}(r,1) \leqslant \mathcal{H}(r_n, q_1 \tau) \leqslant \left(\frac{C 2^{2sn}}{(R-r)^{2s}} \right)^{\tau} \mathcal{H}(r_{n-1}, q_1) \leqslant \mathcal{H}(r_0, q_n) \prod_{j=1}^{n} \left(\frac{C 2^{2s(n-j+1)}}{(R-r)^{2s}} \right)^{\tau^j},$$

where $C = C(N, s, \gamma)$. Moreover, since $q \in (0, \tau^{-1})$ and therefore $q_n < q$ for every $n \in \mathbb{N}$, by Hölder's inequality we obtain

$$\mathcal{H}(r_0, q_n) \leqslant |Q_+(R)|_{d\mu \times dt}^{\frac{1}{q_n} - \frac{1}{q}} \mathcal{H}(R, q) \leqslant |Q_+(1)|_{d\mu \times dt}^{\frac{1}{q_n} - \frac{1}{q}} \mathcal{H}(R, q).$$

Using the estimates detailed in the proof of [96, Theorem 3.7] we conclude the result for $s \geqslant \frac{1}{2}$. Likewise, for $s < \frac{1}{2}$, we define

$$\tilde{r}_j := \left(r^{2s} + \frac{R^{2s} - r^{2s}}{2^j} \right)^{1/2s},$$

and we conclude again as in [96, Theorem 3.7]. $\square$

In order to apply Lemma 2.2.16, our next goal will be to establish bounds for $\log v$. Previously, we need the following auxiliary result.

Lemma 2.2.21. *Let $I \subset \mathbb{R}$ and $\psi : \mathbb{R}^N \to [0,+\infty)$ be a continuous function satisfying $supp(\psi) = \bar{B_R}$ for some $R > 0$ and $|||\psi|||_{Y_0^{s,\gamma}(\mathbb{R}^N)} \leqslant C$. Then, for $v : \mathbb{R}^N \times I \to [0,+\infty)$, the following inequality holds,*

$$\iint_{\mathbb{R}^{2N}} (v(x,t)-v(y,t))(-\psi^2(x)v^{-1}(x,t)+\psi^2(y)v^{-1}(y,t))\,d\nu$$
$$\geqslant \iint_{B_R\times B_R} \psi(x)\psi(y)\left(\log\frac{v(y,t)}{\psi(y)} - \log\frac{v(x,t)}{\psi(x)}\right)^2 d\nu - 3\iint_{\mathbb{R}^{2N}} (\psi(x)-\psi(y))^2\,d\nu.$$

The proof of this result relies on two pointwise inequalities, so it is essentially the same as in the case of the Lebesgue measure ([96, Lemma 4.1]). For the reader convenience, we include here some details.

Proof. First of all, notice that, since ψ vanishes in $\mathbb{R}^N \setminus B_R$, we can split

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} (v(x,t)-v(y,t))(-\psi^2(x)v^{-1}(x,t)+\psi^2(y)v^{-1}(y,t))\,d\nu \\
&\qquad = \int_{B_R}\int_{B_R} (v(x,t)-v(y,t))(-\psi^2(x)v^{-1}(x,t)+\psi^2(y)v^{-1}(y,t))\,d\nu \\
&\qquad\quad + 2\int_{\mathbb{R}^N\setminus B_R}\int_{B_R} (v(x,t)-v(y,t))(-\psi^2(x)v^{-1}(x,t)+\psi^2(y)v^{-1}(y,t))\,d\nu.
\end{aligned} \tag{2.2.31}$$

Fix now $t \in I$ and suppose $x, y \in B_R$, i.e., $\psi(x) \neq 0$, $\psi(y) \neq 0$. Thus, multiplying and dividing by $\psi(x)\psi(y)$ one has

$$\begin{aligned}
&(v(x,t)-v(y,t))(-\psi^2(x)v^{-1}(x,t)+\psi^2(y)v^{-1}(y,t)) \\
&\qquad = \psi(x)\psi(y)\left(\frac{v(y,t)\psi(x)}{\psi(y)v(x,t)} + \frac{v(x,t)\psi(y)}{\psi(x)v(y,t)} - \frac{\psi(y)}{\psi(x)} - \frac{\psi(x)}{\psi(y)}\right).
\end{aligned} \tag{2.2.32}$$

Moreover, applying the numerical inequality

$$\frac{(a-b)^2}{ab} = (a-b)(b^{-1}-a^{-1}) \geqslant (\log a - \log b)^2, \qquad \text{for } a, b > 0,$$

we obtain

$$\begin{aligned}
&\frac{v(y,t)\psi(x)}{\psi(y)v(x,t)} + \frac{v(x,t)\psi(y)}{\psi(x)v(y,t)} - \frac{\psi(y)}{\psi(x)} - \frac{\psi(x)}{\psi(y)} \\
&\qquad \geqslant \left(\log\frac{v(y,t)}{\psi(y)} - \log\frac{v(x,t)}{\psi(x)}\right)^2 - \left(\frac{\psi(x)}{\psi(y)} + \frac{\psi(y)}{\psi(x)} - 2\right).
\end{aligned} \tag{2.2.33}$$

On the other hand, since $v \geqslant 0$ in $\mathbb{R}^N$,

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_R}\int_{B_R}(v(x,t)-v(y,t))(-\psi^2(x)v^{-1}(x,t)+\psi^2(y)v^{-1}(y,t))\,d\nu \\
&\qquad= \int_{\mathbb{R}^N\setminus B_R}\int_{B_R}(v(x,t)-v(y,t))(-\psi^2(x)v^{-1}(x,t))\,d\nu \\
&\qquad\geqslant -\int_{\mathbb{R}^N\setminus B_R}\int_{B_R}\psi^2(x)\,d\nu \\
&\qquad= -\int_{\mathbb{R}^N\setminus B_R}\int_{B_R}(\psi(x)-\psi(y))^2\,d\nu \\
&\qquad\geqslant -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\psi(x)-\psi(y))^2\,d\nu.
\end{aligned} \tag{2.2.34}$$

Plugging (2.2.32), (2.2.33) and (2.2.34) into (2.2.31) we conclude the proof. □

With this result, we can already estimate $\log v$ (see [96, Proposition 4.2] for more details in the case of the Lebesgue measure).

Lemma 2.2.22. *Assume that v is a supersolution to (2.2.6) in the cylinder*

$$Q := B_2 \times (-1,1).$$

Then there exists a positive constant $C = C(N,s)$ such that for some constant $a = a(v)$, we have

$$\forall m>0: \quad |Q_+(1)\cap\{\log v < -m-a\}|_{d\mu\times dt} \leqslant \frac{C|B_1|_{d\mu}}{m}, \tag{2.2.35}$$

$$\forall m>0: \quad |Q_-(1)\cap\{\log v > m-a\}|_{d\mu\times dt} \leqslant \frac{C|B_1|_{d\mu}}{m}. \tag{2.2.36}$$

Proof. Suppose that $v \geqslant \varepsilon > 0$ in Q. Let ψ be such that

$$\psi^2 = \max\left\{\min\left\{\frac{3}{2}-|x|,1\right\},0\right\},$$

and let us denote

$$w(x,t) := -\log\frac{v(x,t)}{\psi(x)}.$$

Using $\dfrac{\psi^2}{v}$ as a test function in (2.2.6) and noticing that $\operatorname{supp}(\psi^2) \subseteq B_{3/2}$, by Lemma 2.2.21 and the fact that $|||\psi|||_{Y_0^{s,\gamma}(\mathbb{R}^N)} \leqslant C$, there follows

$$\begin{aligned}
&\int_{B_{3/2}}\psi^2 w_t d\mu + \frac{a_{N,s}}{2}\iint_{B_{3/2}\times B_{3/2}}\psi(x)\psi(y)(w(x,t)-w(y,t))^2 d\nu \\
&\qquad\leqslant 3\frac{a_{N,s}}{2}\iint_{\mathbb{R}^{2N}}(\psi(x)-\psi(y))^2 d\nu \leqslant C.
\end{aligned}$$

We set $W(t) := \dfrac{\int_{B_{3/2}} \psi^2 w(x,t)d\mu}{\int_{B_{3/2}} \psi^2 d\mu}$, then by the Poincaré type inequality obtained in Theorem 2.2.14, we reach that

$$\int_{B_{3/2}} \psi^2 w_t d\mu + C \int_{B_{3/2}} (w(x,t) - W(t))^2 \psi(x)^2 d\mu \leqslant C.$$

Let $(t_1, t_2) \subset (-1,1)$. Integrating in time the previous inequality, dividing by $\displaystyle\int_{B_{3/2}} \psi^2 \, d\mu$, and noticing that

$$\int_{B_{3/2}} \psi^2 \, d\mu \leqslant 2^{N-2\gamma} |B_1|_{d\mu},$$

one gets

$$\frac{W(t_2) - W(t_1)}{t_2 - t_1} + \frac{C_1}{|B_1|_{d\mu}(t_2 - t_1)} \int_{t_1}^{t_2} \int_{B_1} (w(x,t) - W(t))^2 d\mu \leqslant C_2.$$

Suppose that W is differentiable. Thus, letting $t_2 \to t_1$, we get

$$W'(t) + \frac{C_1}{|B_1|_{d\mu}} \int_{B_1} (w(x,t) - W(t))^2 d\mu \leqslant C_2 \qquad a.e \text{ in } (-1,1). \tag{2.2.37}$$

Defining $\tilde{W}(t) = W(t) - C_2 t$ and $\tilde{w}(x,t) = w(x,t) - C_2 t$, from (2.2.37) we deduce

$$\tilde{W}'(t) + \frac{C_1}{|B_1|_{d\mu}} \int_{B_1} (\tilde{w}(x,t) - \tilde{W}(t))^2 d\mu \leqslant 0 \qquad a.e \text{ in } (-1,1). \tag{2.2.38}$$

Notice that from (2.2.38) we know $\tilde{W}'(t) \leqslant 0$, and therefore

$$\tilde{W}(t) \leqslant W(0) =: a(v) = a \text{ for all } t \in (0,1). \tag{2.2.39}$$

Let $t \in (0,1)$. If we define

$$G_m^+(t) := \{x \in B_1(0) : \tilde{w}(x,t) > m + a\},$$

then for $x \in G_m^+(t)$, we have

$$\tilde{w}(x,t) - \tilde{W}(t) \geqslant m + a - \tilde{W}(t) > 0.$$

Applying this in (2.2.38), it yields

$$\tilde{W}'(t) + \frac{C_1}{|B_1|_{d\mu}} |G_m^+(t)|_{d\mu} (m + a - \tilde{W}(t))^2 \leqslant 0,$$

and hence

$$\frac{-\tilde{W}'(t)}{(m + a - \tilde{W}(t))^2} \geqslant \frac{C_1}{|B_1|_{d\mu}} |G_m^+(t)|_{d\mu}.$$

Integrating this differential inequality over $t \in (0,1)$ we get

$$\begin{aligned}\frac{1}{m} \geqslant \left[\frac{1}{m+a-\tilde{W}(t)}\right]_{t=0}^{1} &\geqslant \frac{C_1}{|B_1|_{d\mu}} \int_0^1 |G_m^+(t)|_{d\mu}\, dt \\ &= \frac{C_1}{|B_1|_{d\mu}} \int_0^1 \int_{\{x \in B_1(0) : \tilde{w}(x,t) > m+a\}} d\mu\, dt \\ &= C_1 \frac{|Q_+(1) \cap \{\tilde{w} > m+a\}|_{d\mu \times dt}}{|B_1|_{d\mu}}.\end{aligned}$$

Since $\psi = 1$ in B_1, thus

$$\tilde{w}(x,t) = w(x,t) - C_2 t = -\log v(x,t) - C_2 t \text{ in } Q_+(1),$$

and therefore

$$|Q_+(1) \cap \{\log v + C_2 t < -m-a\}|_{d\mu \times dt} \leqslant \frac{|B_1|_{d\mu}}{C_1 m}.$$

Thus,

$$\begin{aligned}&|Q_+(1) \cap \{\log v < -m-a\}|_{d\mu \times dt} \\ &\quad \leqslant |Q_+(1) \cap \{\log v + C_2 t < -m-a\}|_{d\mu \times dt} + |Q_+(1) \cap \{C_2 t > m/2\}|_{d\mu \times dt} \\ &\quad \leqslant \frac{2}{C_1 m}|B_1|_{d\mu} + \int_{\frac{m}{2C_2}}^1 \int_{B_1} d\mu\, dt = \frac{2}{C_1 m}|B_1|_{d\mu} + \left(1 - \frac{m}{2C_2}\right)|B_1|_{d\mu} \leqslant \frac{C|B_1|_{d\mu}}{m},\end{aligned}$$

what finishes the proof of (2.2.35).

Likewise, considering

$$G_m^-(t) := \{x \in B_1(0) : \tilde{w}(x,t) < -m+a\}$$

instead of $G_m^+(t)$, and repeating the same procedure, we prove (2.2.36).

When W is not differentiable, the result follows by performing a discretization in time, in the same ways as it is done in [96, Proposition 4.2]. □

We are able now to prove the weighted weak Harnack inequality.

Proof of Theorem 2.2.15. Roughly speaking, the key to prove this result will be to define appropriate functions and parameters so that we can deduce the result from Lemma 2.2.16. Indeed, we divide the proof in two cases. Let $0 < r < 1$ such that $B_r \subset \Omega$.

1. Assume first that $s \geqslant \frac{1}{2}$.

   We set $\theta_1 = \theta_2 = \frac{1}{2}$ and define $U_1(r) = B_r \times (1 - r^{2s}, 1)$, $U_2(r) = B_r \times (-1, -1 + r^{2s})$. In the same way we consider $U_1(1) = Q_+(1)$ and $U_2(1) = Q_-(1)$.

   Let $w_1 := e^{-a} v^{-1}, w_2 := e^a v$ where $a = a(v)$ was defined in (2.2.39). From Lemma 2.2.22 we obtain

$$\begin{aligned}|Q_+(1) \cap \{\log w_1 > m\}|_{d\mu \times dt} &\leqslant \frac{C|B_1|_{d\mu}}{m}, \\ |Q_-(1) \cap \{\log w_2 > m\}|_{d\mu \times dt} &\leqslant \frac{C|B_1|_{d\mu}}{m}.\end{aligned}$$

Using Lemma 2.2.18, it follows that $(w_1, U_1(r))$ satisfies the conditions of Lemma 2.2.16 with $p_0 = \infty$ and η any positive constant. Moreover, by Lemma 2.2.20, $(w_2, U_2(r))$ satisfies the same conditions with $p_0 = 1$ and $\eta = \frac{N}{N+2s} < 1$. Hence we conclude that

$$e^{-a} \sup_{U_1(\frac{1}{2})} v^{-1} = \sup_{U_1(\frac{1}{2})} w_1 \leqslant C \quad \text{and} \quad e^{a} \|v\|_{L^1(U_2(\frac{1}{2}), d\mu)} = \|w_2\|_{L^1(U_2(\frac{1}{2}), d\mu)} \leqslant \tilde{C}.$$

Multiplying both inequalities,

$$\|v\|_{L^1(U_2(\frac{1}{2}), d\mu)} \leqslant C \inf_{U_1(\frac{1}{2})} v,$$

and the result follows in this case.

2. If $0 < s < \frac{1}{2}$, we have to change the domains by setting $\theta_1 = \theta_2 = (\frac{1}{2})^{2s}$ and $U_1(r) = B_{r^{\frac{1}{2s}}} \times (1-r, 1)$, $U_2(r) = B_{r^{\frac{1}{2s}}} \times (-1, -1+r)$. Then the same arguments as in the previous case allow us to conclude.

□

Finally, to end this section, we can establish a boundedness condition on the solutions of (2.2.6).

Proposition 2.2.23. *Let $v \in L^2(T_1, T_2; Y_0^{s,\gamma}(\mathbb{R}^N)) \cap \mathcal{C}([T_1, T_2); L^2(\mathbb{R}^N, |x|^{-\gamma}))$ be a solution to* (2.2.6). *If $f \in L^\infty(T_1, T_2; L^\infty(\Omega))$ and $u_0 \in L^\infty(\Omega)$ then $v \in L^\infty(T_1, T_2; L^\infty(\Omega))$.*

The proof of this Proposition makes use of the well-known numerical iteration result of Stampacchia (see [161]), that can be stated as follows.

Lemma 2.2.24. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function such that*

$$\psi(h) \leqslant \frac{C\psi(k)^\delta}{(h-k)^\beta}, \qquad \text{for all } h > k > 0,$$

where $C > 0$, $\delta > 1$ and $\beta > 0$. Then $\psi(d) = 0$, where $d^\beta = C\psi(0)^{\delta-1} 2^{\frac{\delta\beta}{\delta-1}}$.

Now we can prove Proposition 2.2.23.

Proof. Let $T_1 < \tau < T_2$ to be fixed later, and let us test in (2.2.6) with $G_k(v)$, whose definition can be found in (1.1.8), in $\Omega \times (T_1, \tau)$. Notice that, by Proposition 2.2.4, $G_k(v(\cdot, t)) \in Y_0^{s,\gamma}(\Omega)$. Thus,

(2.2.40)
$$\int_{T_1}^{\tau} \int_\Omega v_t G_k(v)\, d\mu\, dt + \frac{a_{N,s}}{2} \int_{T_1}^{\tau} \iint_{\mathbb{R}^{2N}} (v(x,t) - v(y,t))(G_k(v(x,t)) - G_k(v(y,t)))\, d\nu\, dt$$
$$= \int_{T_1}^{\tau} \int_{A_k^t} f \frac{G_k(v)}{|x|^\gamma}\, dx\, dt,$$

where $A_k^t = A_k^t(v) := \{x \in \Omega : v(x,t) > k\}$. By [134, Lemma 4], one can prove that, for fixed t,

$$(v(x,t) - v(y,t))(G_k(v(x,t)) - G_k(v(y,t))) \geqslant (G_k(v(x,t)) - G_k(v(y,t))^2,$$

and therefore

$$\begin{aligned}\int_{T_1}^{\tau}\iint_{\mathbb{R}^{2N}} &(v(x,t) - v(y,t))(G_k(v(x,t)) - G_k(v(y,t)))\, d\nu\, dt \\ &\geqslant \int_{T_1}^{\tau} \|G_k(v)\|^2_{Y_0^{s,\gamma}(\Omega)}\, dt = \|G_k(v)\|^2_{L^2(T_1,\tau;Y_0^{s,\gamma}(\Omega))}.\end{aligned} \tag{2.2.41}$$

Furthermore,

$$\begin{aligned}\int_{T_1}^{\tau}\int_{\Omega} v_t G_k(v)\, d\mu\, dt &= \int_{T_1}^{\tau}\int_{\Omega} \frac{d}{dt}\int_0^{v(x,s)} G_k(\sigma)\, d\sigma\, d\mu\, dt \\ &= \int_{\Omega}\int_0^{v(x,\tau)} G_k(\sigma)\, d\sigma\, d\mu - \int_{\Omega}\int_0^{v(x,T_1)} G_k(\sigma)\, d\sigma\, d\mu.\end{aligned} \tag{2.2.42}$$

Since $u_0 = v(x,T_1) \in L^\infty(\Omega)$, choosing $k \geqslant \|u_0\|_{L^\infty(\Omega)}$ we have

$$\int_{\Omega}\int_0^{v(x,T_1)} G_k(\sigma)\, d\sigma\, d\mu = 0.$$

Moreover, by the definition of G_k,

$$\int_0^s G_k(\sigma)\, d\sigma = \int_k^s (\sigma - k)\, d\sigma = \left[\frac{(\sigma-k)^2}{2}\right]_{\sigma=k}^{s} = \frac{(s-k)^2}{2} \geqslant \frac{G_k^2(s)}{2},$$

and using these two facts in (2.2.42) we obtain

$$\int_{T_1}^{\tau}\int_{\Omega} v_t G_k(v)\, d\mu\, dt \geqslant \frac{1}{2}\int_{\Omega} G_k^2(v(x,\tau))\, d\mu. \tag{2.2.43}$$

Thus, from (2.2.40), (2.2.41) and (2.2.43) we obtain

$$\begin{aligned}|||G_k(v)|||^2 &:= \|G_k(v)\|^2_{L^\infty(T_1,\tau;L^2(\Omega,|x|^{-\gamma}))} + \|G_k(v)\|^2_{L^2(T_1,\tau;Y_0^{s,\gamma}(\Omega))} \\ &\leqslant C\int_{T_1}^{\tau}\int_{A_k^t} f\frac{G_k(v)}{|x|^\gamma}\, dx.\end{aligned} \tag{2.2.44}$$

We deal now with the right hand side of (2.2.40). If we define

$$\zeta(k) := \int_{T_1}^{\tau} |A_k^t|_{dx}\, dt,$$

where $|A_k^t|_{dx}$ denotes the Lebesgue measure of the set A_k^t, then it is satisfied

$$\begin{aligned}\int_{T_1}^{\tau}\int_{A_k^t} f\frac{G_k(v)}{|x|^\gamma}\, dx\, dt &\leqslant \int_{T_1}^{\tau}\int_{A_k^t} fG_k^2(v)\, d\mu\, dt + \int_{T_1}^{\tau}\int_{A_k^t} f\, dx\, dt \\ &\leqslant \int_{T_1}^{\tau}\int_{A_k^t} fG_k^2(v)\, d\mu\, dt + \|f\|_{L^\infty(T_1,\tau;L^\infty(\Omega))}\zeta(k).\end{aligned} \tag{2.2.45}$$

Let us denote $q := 2(1+\frac{2s}{N})$. Thus, applying Hölder's at every step,

(2.2.46)
$$\begin{aligned}\int_{T_1}^{\tau}\int_{A_k^t} fG_k^2(v)\,d\mu\,dt &\leqslant \|f\|_{L^\infty(T_1,\tau;L^\infty(\Omega))}\int_{T_1}^{\tau}\left(\int_{A_k^t}\frac{G_k^q(v)}{|x|^{q\gamma}}\,dx\right)^{\frac{2}{q}}|A_k^t|_{dx}^{1-\frac{2}{q}}\,dt\\ &\leqslant C\left(\int_{T_1}^{\tau}\|G_k(v)\|^q_{L^q(\Omega,|x|^{-\gamma})}\,dt\right)^{\frac{2}{q}}\left(\int_{T_1}^{\tau}|A_k^t|_{dx}\,dt\right)^{1-\frac{2}{q}}\\ &\leqslant C\zeta(k)^{1-\frac{2}{q}}\left(\int_{T_1}^{\tau}\|G_k(v)\|^q_{L^q(\Omega,|x|^{-\gamma})}\,dt\right)^{\frac{2}{q}}\\ &\leqslant C\zeta(k)^{1-\frac{2}{q}}\left(\int_{T_1}^{\tau}\|G_k(v)\|^2_{L^{2_s^*}(\Omega,|x|^{-\gamma})}\|G_k(v)\|^{\frac{4s}{N}}_{L^2(\Omega,|x|^{-\gamma})}\,dt\right)^{\frac{2}{q}}.\end{aligned}$$

Denoting $\theta := \frac{4s}{2N+4s}$, we can write

$$\frac{4s}{N}=\theta q,\qquad 2=(1-\theta)q,$$

and thus, by Theorem 2.2.6, from (2.2.46) we deduce

(2.2.47)
$$\begin{aligned}\int_{T_1}^{\tau}\int_{A_k^t} fG_k^2(v)\,d\mu\,dt &\leqslant C\zeta(k)^{1-\frac{2}{q}}\|G_k(v)\|^{2\theta}_{L^\infty(T_1,\tau;L^2(\Omega,|x|^{-\gamma}))}\|G_k(v)\|^{(1-\theta)2}_{L^2(T_1,\tau;Y_0^s(\Omega))}\\ &\leqslant C\zeta(k)^{1-\frac{2}{q}}\left(\|G_k(v)\|^2_{L^\infty(T_1,\tau;L^2(\Omega,|x|^{-\gamma}))}+\|G_k(v)\|^2_{L^2(T_1,\tau;Y_0^s(\Omega))}\right)\\ &= C\zeta(k)^{1-\frac{2}{q}}|||G_k(v)|||^2,\end{aligned}$$

where in the second step we have used Young's inequality with exponents $p_1 := \frac{1}{\theta}$ and $p_2 := \frac{1}{1-\theta}$. Thus, from (2.2.44), (2.2.45) and (2.2.47) we conclude

$$|||G_k(v)|||^2 \leqslant C\left(\zeta(k)^{1-\frac{2}{q}}|||G_k(v)|||^2+\zeta(k)\right).$$

Noticing that $\zeta(k)\leqslant \tau|\Omega|_{dx}$, we can fix τ small enough so that

$$\|G_k(v)\|^2_{L^q(T_1,\tau;L^q(\Omega,|x|^{-\gamma}))}\leqslant C|||G_k(v)|||^2\leqslant C\zeta(k). \tag{2.2.48}$$

On the other hand, if we take $z>k$,

$$\begin{aligned}\|G_k(v)\|^2_{L^q(T_1,\tau;L^q(\Omega,|x|^{-\gamma}))} &\geqslant C\|G_k(v)\|^2_{L^q(T_1,\tau;L^q(\Omega))}=C\|G_k(v)\|^2_{L^q(T_1,\tau;L^q(A_k^t))}\\ &\geqslant C\|G_k(v)\|^2_{L^q(T_1,\tau;L^q(A_z^t))}\geqslant C(z-k)^2\zeta(z)^{\frac{2}{q}},\end{aligned} \tag{2.2.49}$$

and thus, putting (2.2.48) and (2.2.49) together, we obtain

$$\zeta(z)\leqslant \frac{C}{(z-k)^q}\zeta(k)^{\frac{q}{2}}.$$

Hence, noticing that $\frac{q}{2} > 1$, Lemma 2.2.24 ensures that there exists d such that $\zeta(d) = 0$, that is, $|A_d^t|_{dx} = 0$ for every $t \in (T_1, \tau)$, and therefore necessarily $v \in L^\infty(T_1, \tau; L^\infty(\Omega))$.

Iterating this argument in $[\tau, 2\tau], \ldots, [j\tau, (j+1)\tau]$ to cover the whole range (T_1, T_2), we conclude that $v \in L^\infty(T_1, T_2; L^\infty(\Omega))$. □

2.3 The linear problem: dependence on the spectral parameter λ.

Along this section we will study the problem

$$\begin{cases} u_t + (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + g(x,t) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) = u_0(x) \gneqq 0 & \text{if } x \in \Omega, \end{cases} \tag{2.3.1}$$

where $g(x,t)$ is a nonnegative function. The goal will be to establish some necessary and sufficient conditions on g and u_0 in order to find solutions of this problem. Indeed, the relevant fact here is the spectral dependence of these summability conditions. These results correspond to the ones obtained by P. Baras and J. A. Goldstein in [30] for the classical heat equation with the Hardy potential.

First, we deal with the necessary summability conditions on g and u_0.

Theorem 2.3.1. *Let $0 < \lambda \leqslant \Lambda_{N,s}$. Assume that there exists a positive weak supersolution $\tilde{u}$ to the problem* (2.3.1). *Then g and u_0 must satisfy*

$$\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\gamma} g \, dx \, dt < +\infty, \qquad \int_{B_r(0)} |x|^{-\gamma} u_0 \, dx < +\infty,$$

for any cylinder $B_r(0) \times (t_1, t_2) \subset\subset \Omega \times (0,T)$, where γ was defined in (1.2.3).

Proof. Fix $\varepsilon > 0$. Let consider φ_n, the positive solution to

$$\begin{cases} -(\varphi_n)_t + (-\Delta)^s \varphi_n = \lambda \dfrac{\varphi_{n-1}}{|x|^{2s} + \frac{1}{n}} + 1 & \text{in } \Omega \times (-\varepsilon, T), \\ \varphi_n = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (-\varepsilon, T], \\ \varphi_n(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{2.3.2}$$

with

$$\begin{cases} -(\varphi_0)_t + (-\Delta)^s \varphi_0 = 1 & \text{in } \Omega \times (-\varepsilon, T), \\ \varphi_0 = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (-\varepsilon, T], \\ \varphi_0(x,T) = 0 & \text{in } \Omega. \end{cases}$$

By comparison, $\varphi_{n-1} \leqslant \varphi_n \leqslant \varphi$, where φ is the positive energy solution to

$$\begin{cases} -\varphi_t + (-\Delta)^s \varphi - \lambda \dfrac{\varphi}{|x|^{2s}} = 1 & \text{in } \Omega \times (-\varepsilon, T), \\ \varphi = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (-\varepsilon, T], \\ \varphi(x,T) = 0, & \text{in } \Omega. \end{cases}$$

Thus, if we define $\phi(x,t) := |x|^{-\gamma}\varphi(x,t)$, then $\phi \in L^2(-\varepsilon, T; Y^{s,\gamma}(\mathbb{R}^N))$ and it solves the problem

$$
\left\{
\begin{array}{rl}
-|x|^{2\gamma}\phi_t + L_\gamma \phi = |x|^{-\gamma} & \text{in } \Omega \times (-\varepsilon, T),\\
\phi = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (-\varepsilon, T],\\
\phi(x,T) = 0, & \text{in } \Omega,
\end{array}
\right.
\tag{2.3.3}
$$

where L_γ and $Y^{s,\gamma}$ were defined in (2.2.4) and (2.2.8) respectively. Hence, as a consequence of Theorem 2.2.15, for any cylinder $B_r(0) \times (t_1, t_2) \subset\subset \Omega \times (-\varepsilon, T)$ there exists a constant $A > 0$ such that $\phi \geqslant A$ in $B_r(0) \times (t_1, t_2)$, i.e.,

$$
\varphi(x,t) \geqslant \frac{A}{|x|^\gamma}, \qquad (x,t) \in B_r(0) \times (t_1, t_2).
\tag{2.3.4}
$$

Moreover, since φ_n is regular and bounded we can use it as a test function in (2.3.1), but also as a pointwise solution of (2.3.2). Thus, noticing that $(-\Delta)^s\varphi \leqslant 0$ for $x \in \mathbb{R}^N \setminus \Omega$, we obtain

$$
\begin{aligned}
&\int_0^T \int_\Omega g\varphi_n \, dx\, dt + \int_\Omega u_0 \varphi_n(x,0)\, dx \\
&\quad \leqslant -\int_0^T \int_\Omega \tilde{u}(\varphi_n)_t \, dx\, dt + \int_0^T \int_{\mathbb{R}^N} \tilde{u}(-\Delta)^s \varphi_n \, dx\, dt - \lambda \int_0^T \int_\Omega \frac{\tilde{u}\varphi_n}{|x|^{2s}} \, dx\, dt \\
&\quad \leqslant -\int_0^T \int_\Omega \tilde{u}(\varphi_n)_t \, dx\, dt + \int_0^T \int_\Omega \tilde{u}(-\Delta)^s \varphi_n \, dx\, dt - \lambda \int_0^T \int_\Omega \frac{\tilde{u}\varphi_{n-1}}{|x|^{2s} + \frac{1}{n}} \, dx\, dt \\
&\quad = \int_0^T \int_\Omega \tilde{u}\, dx\, dt = C < +\infty.
\end{aligned}
\tag{2.3.5}
$$

Since both integrals in the left hand side are positive, in particular each one is uniformly bounded. Thus, by the Monotone Convergence Theorem and (2.3.5),

$$
\begin{aligned}
A\int_{t_1}^{t_2} \int_{B_r(0)} |x|^{-\gamma} g \, dx dt &\leqslant \int_{t_1}^{t_2} \int_{B_r(0)} g\varphi \, dx\, dt \\
&\leqslant \int_0^T \int_\Omega g\varphi\, dx\, dt = \lim_{n\to\infty} \int_0^T \int_\Omega g\varphi_n \, dx\, dt < +\infty.
\end{aligned}
$$

Likewise, $\{u_0(\cdot)\varphi_n(\cdot, 0)\}$ is also an increasing sequence, uniformly bounded in $L^1(\Omega)$, and thus, choosing t_1 and t_2 so that $0 \in (t_1, t_2) \subsetneq (-\varepsilon, T)$, as above we conclude

$$
A\int_{B_r(0)} |x|^{-\gamma} u_0(x)\, dx \leqslant \int_\Omega u_0(x)\varphi(x,0)\, dx < +\infty.
$$

$\square$

Conversely, we would like to find the optimal summability conditions on g and u_0 to prove existence of weak solution. In this direction, notice that if $g \in L^2(0, T; H^{-s}(\Omega))$ and $u_0 \in L^2(\Omega)$, by Remark 2.1.5 we can assure the existence of an energy solution to (2.3.1) whether $\lambda < \Lambda_{N,s}$, and also for $\lambda = \Lambda_{N,s}$ in a weaker sense. A sharper result, for a more general class of data, is the following.

Theorem 2.3.2. *Assume* $0 < \lambda < \Lambda_{N,s}$, *and that* g *and* u_0 *satisfy*

$$\int_\Omega |x|^{-\gamma} u_0 \, dx < +\infty, \qquad \int_0^T \int_\Omega |x|^{-\gamma} g dx \, dt < +\infty,$$

where γ *was defined in* (1.2.3). *Then problem* (2.3.1) *has a positive weak solution.*

Proof. Consider the approximated problems

$$\begin{cases} u_{nt} + (-\Delta)^s u_n = \lambda \dfrac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} + g_n & \text{in } \Omega \times (0,T), \\ u_n(x,t) > 0 & \text{in } \Omega \times (0,T), \\ u_n(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u_n(x,0) = T_n(u_0(x)) & \text{if } x \in \Omega, \end{cases} \tag{2.3.6}$$

where

$$\begin{cases} u_{0t} + (-\Delta)^s u_0 = g_1 & \text{in } \Omega \times (0,T), \\ u_0(x,t) > 0 & \text{in } \Omega \times (0,T), \\ u_0(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u_0(x,0) = T_1(u_0(x)) & \text{if } x \in \Omega, \end{cases} \tag{2.3.7}$$

and

$$g_n := T_n(g) = \begin{cases} g & \text{if } |g| \leqslant n, \\ n\dfrac{g}{|g|} & \text{if } |g| > n. \end{cases}$$

By Lemma 2.1.9, it follows that $u_0 \leqslant u_1 \leqslant \cdots \leqslant u_{n-1} \leqslant u_n$ in $\mathbb{R}^N \times (0,T)$. Note that, since the right hand sides of these problems are bounded, every u_n is actually an energy solution.

Consider φ the energy solution of the problem

$$\begin{cases} -\varphi_t + (-\Delta)^s \varphi - \lambda \dfrac{\varphi}{|x|^{2s}} = 1 & \text{in } \Omega \times (-\varepsilon,T), \\ \varphi > 0 & \text{in } \Omega \times (-\varepsilon,T), \\ \varphi = 0 & \text{on } (\mathbb{R}^N \setminus \Omega) \times (-\varepsilon,T], \\ \varphi(x,T) = \varphi_T & \text{in } \Omega, \end{cases} \tag{2.3.8}$$

where φ_T is a positive constant. Proceeding as in the proof of Theorem 2.3.1, we consider $\phi := |x|^\gamma \varphi$, that solves the problem (2.3.3) with $\phi(x,T) = |x|^\gamma \varphi_T$ in Ω. Thus, by Proposition 2.2.23, $\phi \in L^\infty(-\varepsilon, T; L^\infty(\Omega))$, that is, there exists $C > 0$ such that

$$\varphi(x,t) \leqslant \frac{C}{|x|^\gamma}, \text{ in } \Omega \times (-\varepsilon, T). \tag{2.3.9}$$

Since φ also belongs to $L^2(0,T;H_0^s(\Omega))$, we can use it as a test function in (2.3.6). Thus,

$$\begin{aligned}\int_0^T\int_\Omega (u_n)_t\varphi\,dx\,dt+\int_0^T\iint_{\mathbb{R}^{2N}}\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}\,dxdy\,dt\\ =\lambda\int_0^T\int_\Omega\frac{u_{n-1}\varphi}{|x|^{2s}+\frac{1}{n}}\,dx\,dt+\int_0^T\int_\Omega g_n\varphi\,dx\,dt\\ \leqslant\lambda\int_0^T\int_\Omega\frac{u_n\varphi}{|x|^{2s}}\,dx\,dt+\int_0^T\int_\Omega g_n\varphi\,dx\,dt.\end{aligned}$$

Thinking now of the left hand side as problem (2.3.8) tested with u_n, and applying (2.3.9), we conclude that

$$\begin{aligned}\varphi_T\int_\Omega u_n(x,T)dx+\int_0^T\int_\Omega u_n\,dx\,dt &\leqslant\int_0^T\int_\Omega g_n\varphi\,dx\,dt+\int_\Omega T_n(u_0(x))\varphi(x,0)\,dx\\ &\leqslant\int_0^T\int_\Omega g\varphi\,dx\,dt+\int_\Omega u_0(x)\varphi(x,0)\,dx\\ &\leqslant C\int_0^T\int_\Omega\frac{g}{|x|^\gamma}\,dx\,dt+C\int_\Omega\frac{u_0(x)}{|x|^\gamma}\,dx\\ &<+\infty,\end{aligned}$$

by hypotheses. Hence, since the sequence $\{u_n\}$ is increasing, we can define $u:=\lim_{n\to\infty}u_n$, and conclude that $u\in L^1(\Omega\times(0,T))$ by applying the Monotone Convergence Theorem.

Notice that, using the same computations as above and integrating in $\Omega\times[0,t]$ with $t\leqslant T$, by considering the above estimates on $\{u_n\}_{n\in\mathbb{N}}$, we reach that

$$\sup_{t\in[0,T]}\int_\Omega u_n(x,t)dx+\int_0^T\int_\Omega u_n\,dx\,dt\leqslant C\ \text{ for all } n. \tag{2.3.10}$$

Fix $T_1>T$, and define $\tilde\varphi$ as the unique solution to the problem

$$\begin{cases}-\tilde\varphi_t+(-\Delta)^s\tilde\varphi=1 & \text{in }\Omega\times(-\varepsilon,T_1),\\ \tilde\varphi>0 & \text{in }\Omega\times(-\varepsilon,T_1),\\ \tilde\varphi=0 & \text{on }(\mathbb{R}^N\setminus\Omega)\times(-\varepsilon,T_1],\\ \varphi(x,T_1)=0 & \text{in }\Omega.\end{cases} \tag{2.3.11}$$

In particular, $\tilde\varphi\in L^\infty(\Omega\times(-\varepsilon,T_1))$ (see for example [134, Corollary 3]), and by Theorem 2.1.11 $\tilde\varphi(x,t)\geqslant\bar C>0$ for all $(x,t)\in B_r(0)\times[0,T]$, where $B_r(0)\subset\subset\Omega$. Now, using $\tilde\varphi$ as a test function in (2.3.6) and integrating in $\Omega\times(0,T)$, it follows that

$$\int_\Omega u_n(x,T)\tilde\varphi(x,T)\,dx\,dt+\int_0^T\int_\Omega u_n\,dx\,dt\geqslant\lambda\int_0^T\int_\Omega\frac{u_{n-1}\tilde\varphi}{|x|^{2s}+\frac{1}{n}}\,dx\,dt.$$

Thus, by (2.3.10),

$$\lambda\int_0^T\int_\Omega\frac{u_{n-1}\tilde\varphi}{|x|^{2s}+\frac{1}{n}}\,dx\,dt\leqslant C\sup_{\{t\in[0,T]\}}\int_\Omega u_n(x,t)dx+\int_0^T\int_\Omega u_n\,dx\,dt\leqslant C\quad\text{for all}\quad n.$$

Hence

$$\begin{aligned}\int_0^T \int_\Omega \frac{u_{n-1}}{|x|^{2s}+\frac{1}{n}}\,dx\,dt &= \int_0^T \int_{B_r(0)} \frac{u_{n-1}}{|x|^{2s}+\frac{1}{n}}\,dx\,dt + \int_0^T \int_{\Omega\setminus B_r(0)} \frac{u_{n-1}}{|x|^{2s}+\frac{1}{n}}\,dx\,dt\\ &\leqslant C\int_0^T \int_\Omega \frac{u_{n-1}\tilde{\varphi}}{|x|^{2s}+\frac{1}{n}}\,dx\,dt + C\int_0^T \int_\Omega u_{n-1}\,dx\,dt \leqslant C.\end{aligned}$$

Therefore, by the Monotone Convergence Theorem we conclude that

$$\frac{u_{n-1}}{|x|^{2s}+\frac{1}{n}} + g_n \nearrow \frac{u}{|x|^{2s}} + g \text{ in } L^1(\Omega\times(0,T)).$$

To conclude that u is a weak solution to problem (2.3.1), it remains to check that u belongs to $\mathcal{C}([0,T);L^1(\Omega))$. We claim that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T];L^1(\Omega))$, and hence the result follows. In order to prove this, we follow the arguments in [148].

Let $n,m\in\mathbb{N}$, such that $n\geqslant m$, and denote $u_{n,m}:=u_n-u_m$, and $g_{n,m}:=g_n-g_m$. Clearly, $u_{n,m}$, $g_{n,m}\geqslant 0$. We set

$$C_{n,m} := \lambda\int_0^T\int_\Omega \frac{u_{n,m}}{|x|^{2s}}\,dx\,d\tau + \int_0^T\int_\Omega g_{n,m}\,dx\,d\tau,$$

and then $C_{n,m}\to 0$ as $n,m\to\infty$.

By the definition of the approximated problems in (2.3.6) and the linearity of the operator, for $t\leqslant T$,

$$\int_0^t\int_\Omega (u_{n,m})_t T_1(u_{n,m})\,dx\,d\tau + \int_0^t\int_\Omega (-\Delta)^s(u_{n,m})T_1(u_{n,m})\,dx\,d\tau \leqslant C_{n,m}.$$

Since $u_{n,m}\in L^2((0,T);H_0^s(\Omega))$, it follows that (see [134, Proposition 3] for a detailed proof)

$$\int_0^t\int_\Omega (-\Delta)^s(u_{n,m})T_1(u_{n,m})\,dx\,d\tau \geqslant \int_0^t \|T_1(u_{n,m})\|^2_{H_0^s(\Omega)}\,d\tau \geqslant 0,$$

and therefore,

$$\int_0^t\int_\Omega (u_{n,m})_t T_1(u_{n,m})\,dx\,d\tau \leqslant C_{n,m}.$$

Let define

$$\Psi(s) := \int_0^s T_1(\sigma)d\sigma.$$

Since $u_n\in\mathcal{C}([0,T];L^2(\Omega))$, then

$$\int_0^t\int_\Omega (u_{n,m})_t T_1(u_{n,m})\,dx\,d\tau = \int_\Omega (\Psi(u_{n,m})(t)-\Psi(u_{n,m})(0))\,dx.$$

Thus

$$\int_\Omega \Psi(u_{n,m})(t)\,dx \leqslant C_{n,m} + \int_\Omega \Psi(u_{n,m})(0)\,dx,$$

where the right hand side is independent of t. Since $\Psi(u_{n,m})(0) = \Psi(T_n(u_0) - T_m(u_0))$, noticing that $\Psi(s) \leqslant s$ and $T_n(u_0) - T_m(u_0) \to 0$ in $L^1(\Omega)$ as $n, m \to \infty$, we obtain that

$$\int_\Omega \Psi(u_{n,m})(t)\, dx \to 0 \text{ as } n, m \to \infty, \text{ uniformly in } t. \tag{2.3.12}$$

Now, since

$$\int_{|u_{n,m}|<1} \frac{|u_{n,m}|^2(t)}{2}\, dx + \int_{|u_{n,m}|>1} |u_{n,m}|(t)\, dx \leqslant \int_\Omega \Psi(u_{n,m})(t)\, dx,$$

by (2.3.12) we know that $u_{n,m} \to 0$ uniformly in t. Thus $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T]; L^1(\Omega))$ and passing to the limit in the weak formulation of the approximated problems, one obtains that u is a positive weak solution of problem (2.3.1) in $\Omega\times(0,T)$. □

Remark 2.3.3. *Since γ depends on λ, Theorem 2.3.1 and Theorem 2.3.2 illustrate the dependence of the summability of the data on the spectral parameter λ.*

Next, we see that $\Lambda_{N,s}$ provides an actual restriction on λ.

Proposition 2.3.4. *If $\lambda > \Lambda_{N,s}$, problem* (2.3.1) *has no positive weak supersolution.*

Proof. Notice first that we can write problem (2.3.1) as

$$\begin{cases} u_t + (-\Delta)^s u - \Lambda_{N,s} \dfrac{u}{|x|^{2s}} = (\lambda - \Lambda_{N,s}) \dfrac{u}{|x|^{2s}} + g & \text{in } \Omega \times (0,T), \\ u(x,t) > 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T). \end{cases} \tag{2.3.13}$$

Thus, if $\tilde{u}$ is a supersolution of (2.3.1), then it is so also of (2.3.13). Moreover, since in the left hand side of this problem the constant is $\Lambda_{N,s}$, this matches with the case $\alpha = 0$ in Lemma 1.2.3 and hence, by Theorem 2.3.1, necessarily

$$\left((\lambda - \Lambda_{N,s}) \frac{\tilde{u}}{|x|^{2s}} + g \right) |x|^{-\frac{N-2s}{2}} \in L^1(B_r(0) \times (t_1, t_2)),$$

for any $B_r(0) \times (t_1,t_2) \subset\subset \Omega \times (0,T)$ small enough. In particular, this implies

$$(\lambda - \Lambda_{N,s}) \frac{\tilde{u}}{|x|^{2s}} |x|^{-\frac{N-2s}{2}} \in L^1(B_r(0) \times (t_1, t_2)), \tag{2.3.14}$$

and hence, defining $\tilde{v} := |x|^\gamma \tilde{u}$, we can apply again Theorem 2.2.15 to conclude from (2.3.14) that

$$(\lambda - \Lambda_{N,s}) |x|^{-N} \in L^1(B_r(0) \times (t_1, t_2)),$$

what is a contradiction. Therefore, there does not exist a positive supersolution of problem (2.3.1) if $\lambda > \Lambda_{N,s}$. □

Remark 2.3.5. *The previous nonexistence result implies that for $\lambda > \Lambda_{N,s}$ an instantaneous and complete blow up phenomena occurs. The proof is a simple adaptation of Theorem 2.4.4, where this result will be proved for a more involved semilinear problem.*

Furthermore, we can state a nonexistence result that shows the optimality of the power $p = 1$ in the singular term $\dfrac{u^p}{|x|^{2s}}$. The proof for this nonlocal problem closely follows the classical case, due to H. Brezis and X. Cabré (see [47]).

Theorem 2.3.6. *Let $p > 1$, and let $u \geqslant 0$ satisfy*

$$u_t + (-\Delta)^s u \geqslant \frac{u^p}{|x|^{2s}} \text{ in } \Omega \times (0, T),$$

in the weak sense. Then $u \equiv 0$.

Proof. Consider a cylinder $B_\tau(0) \times (t_1, t_2)$. If $u \gneqq 0$, by the Maximum Principle (Theorem 2.1.12), we know that there exists $\varepsilon > 0$ so that

$$u \geqslant \varepsilon \text{ in } B_\tau(0) \times (t_1, t_2).$$

Let define

$$\phi(s) = \begin{cases} \dfrac{1}{(p-1)\varepsilon^{p-1}} - \dfrac{1}{(p-1)s^{p-1}} & \text{if } s \geqslant \varepsilon, \\ \\ \dfrac{1}{\varepsilon^p}(s - \varepsilon) & \text{if } s < \varepsilon. \end{cases}$$

Notice that $0 \leqslant \phi < +\infty$ in $[\varepsilon, +\infty)$, $\phi(\varepsilon) = 0$, and ϕ is a $\mathcal{C}^1$ function satisfying $\phi'(s) = \frac{1}{s^p}$ for $s \geqslant \varepsilon$. Moreover, since ϕ is concave, it follows that $(-\Delta)^s(\phi(u)) \geqslant \phi'(u)(-\Delta)^s u$ (see for example [134, Proposition 4]) and thus,

$$(\phi(u))_t + (-\Delta)^s(\phi(u)) \geqslant \phi'(u)\,(u_t + (-\Delta)^s u) \geqslant \frac{1}{|x|^{2s}} \text{ for } u \geqslant \varepsilon.$$

Consider now the weak solution w to the elliptic equation

$$\begin{cases} (-\Delta)^s w = \dfrac{1}{|x|^{2s}} & \text{in } B_\tau(0), \\ \qquad w = 0 & \text{on } \mathbb{R}^N \setminus B_\tau(0), \end{cases}$$

that exists because $\frac{1}{|x|^{2s}} \in L^1(B_\tau(0))$ (see [134, Theorem 28]). Hence,

$$(\phi(u) - w)_t + (-\Delta)^s(\phi(u) - w) \geqslant 0 \text{ in } B_\tau(0) \times (t_1, t_2),$$

in the weak sense, what implies $\phi(u) - w \geqslant 0$ in $B_\tau(0) \times (t_1, t_2)$. Thus, if we prove that w is unbounded, we reach a contradiction with the fact that ϕ is bounded, and the proof is finished.

Define

$$\tilde{w} := \frac{1}{|x|^{2s}} * \frac{C_{N,s}}{|x|^{N-2s}},$$

where the constant is chosen such that $\frac{C_{N,s}}{|x|^{N-2s}}$ is the fundamental solution of the fractional Laplace equation. Thus, for $|x| \leqslant 1/n$,

$$\begin{aligned}\tilde{w}(x) &= C_{N,s} \int_{B_\tau(0)} \frac{1}{|y|^{2s}|x-y|^{N-2s}}\, dy \\ &\geqslant \tilde{C} \int_{B_\tau(0)} \frac{1}{|y|^{2s}\left(|y|^{N-2s} + (1/n)^{N-2s}\right)}\, dy \to +\infty\end{aligned}$$

when $n \to +\infty$, that is, when $|x| \to 0$. But recalling the definition of $\tilde{w}$, we have that $w - \tilde{w}$ is s-harmonic in $B_\tau(0)$, and hence bounded in $B_{\tau/2}(0)$. Therefore, $w(x) \to +\infty$ as $|x| \to 0$. $\square$

Thus, as a straightforward consequence we obtain the following result.

Corollary 2.3.7. *Let $g \in L^1(\Omega \times (0,T))$, $g \geqslant 0$, and $p > 1$. Therefore, the problem*

$$\begin{cases} u_t + (-\Delta)^s u = \lambda \dfrac{u^p}{|x|^{2s}} + g & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) = u_0(x) \gneqq 0 & \text{if } x \in \Omega, \end{cases}$$

has no positive weak solution.

2.4 Existence and nonexistence results for a semilinear problem.

The goal of this section is to study how the addition of a semilinear term of the form u^p, with $p > 1$, interferes with the solvability of the previous problems. As in the classical heat equation (see [8]), the relevant feature is that for every $\lambda \in (0, \Lambda_{N,s})$ there exists a threshold for the existence, $p(\lambda, s)$, that depends on the spectral parameter. Indeed, we will consider the problem

$$(2.4.1) \qquad \begin{cases} u_t + (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + u^p + f & \text{in } \Omega \times (0,T), \\ u(x,t) > 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) = u_0(x) & \text{if } x \in \Omega, \end{cases}$$

with $p > 1$ and $0 < \lambda \leqslant \Lambda_{N,s}$. By weak or energy solutions of this problem, we mean solutions in the sense of Definition 2.1.1 and Definition 2.1.2 by fixing $F = \lambda \dfrac{u}{|x|^{2s}} + u^p + cf$.

We will prove that there exists such critical exponent $p(\lambda, s)$ so that one can prove existence of solution for problem (2.4.1) whether $1 < p < p(\lambda, s)$, and nonexistence for $p > p(\lambda, s)$. In particular, we will see that

$$p(\lambda, s) = 1 + \frac{2s}{\gamma},$$

where γ was defined in (1.2.3), is exactly the same threshold that we found in Chapter 1 for the elliptic case. Note that if $\lambda = \Lambda_{N,s}$, namely, $\alpha = 0$, then $p(\lambda, s) = 2_s^* - 1$, and if $\lambda = 0$, i.e., $\alpha = \dfrac{N-2s}{2}$, then $p(\lambda, s) = \infty$.

We will need some auxiliary results that allow us to build a solution whenever we have a supersolution to our problem. To prove existence of a weak solution to (2.4.1) with L^1 data from a weak supersolution, we will consider the *solution obtained as limit of solutions of the approximated problems* (see [41, 75] for the local parabolic operators case).

Lemma 2.4.1. *Assume $f \in L^1(\Omega \times (0,T))$ and $\lambda \leqslant \Lambda_{N,s}$. If $\bar{u} \in \mathcal{C}([0,T]; L^1(\Omega))$ is a weak positive supersolution to the problem* (2.4.1)*, then there exists a positive weak solution to problem* (2.4.1) *obtained as limit of solutions of approximated problems.*

Proof. If $\bar{u}$ is a positive supersolution to (2.4.1) with $\lambda \leqslant \Lambda_{N,s}$, we consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ of energy solutions of the problems

$$\left\{\begin{aligned} u_{nt} + (-\Delta)^s u_n &= \lambda \frac{u_{n-1}}{|x|^{2s} + \frac{1}{n}} + u_{n-1}^p + T_n(f) && \text{in } \Omega \times (0,T),\\ u_n(x,t) &> 0 && \text{in } \Omega \times (0,T),\\ u_n(x,t) &= 0 && \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T),\\ u_n(x,0) &= T_n(u_0(x)) && \text{if } x \in \Omega, \end{aligned}\right. \tag{2.4.2}$$

starting with

$$\left\{\begin{aligned} u_{0t} + (-\Delta)^s u_0 &= T_1(f) && \text{in } \Omega \times (0,T),\\ u_0(x,t) &> 0 && \text{in } \Omega \times (0,T),\\ u_0(x,t) &= 0 && \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T),\\ u_0(x,0) &= T_1(u_0(x)) && \text{if } x \in \Omega. \end{aligned}\right. \tag{2.4.3}$$

By Lemma 2.1.9, it follows $u_0 \leqslant \ldots \leqslant u_{n-1} \leqslant u_n \leqslant \bar{u}$ in $\mathbb{R}^N \times (0,T)$. Since $\bar{u} \in \mathcal{C}([0,T]; L^1(\Omega))$ is a supersolution, by monotonicity we can define the pointwise limit $u := \lim u_n$ that verifies $u \leqslant \bar{u}$, and therefore $u \in L^1(\Omega \times (0,T))$ and

$$\frac{u}{|x|^{2s}} + u^p + f \in L^1(\Omega \times (0,T)).$$

Thus, u satisfies

$$\left\{\begin{aligned} u_t + (-\Delta)^s u &= \lambda \frac{u}{|x|^{2s}} + u^p + f && \text{in } \Omega \times (0,T),\\ u(x,t) &> 0 && \text{in } \Omega \times (0,T),\\ u(x,t) &= 0 && \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T),\\ u(x,0) &= u_0(x) && \text{if } x \in \Omega, \end{aligned}\right. \tag{2.4.4}$$

in the weak sense. To see that indeed $u \in \mathcal{C}([0,T); L^1(\Omega))$ we proceed as in the proof of Theorem 2.3.2. □

Likewise, if the supersolution belongs to the energy space, the solution we find will be also an energy solution.

Lemma 2.4.2. *Let $f \in L^2(0,T; H^{-s}(\Omega))$ and $\lambda < \Lambda_{N,s}$. If $\bar{u} \in L^2(0,T; H_0^s(\Omega))$ with $\bar{u}_t \in L^2(0,T; H^{-s}(\Omega))$ is a positive finite energy supersolution to* (2.4.1) *with $\lambda \leqslant \Lambda_{N,s}$ and $f \in L^2(0,T; H^{-s}(\Omega))$, then there exists a positive energy solution to the problem* (2.4.1)*, obtained as limit of solutions of the approximated problems.*

Proof. Proceeding as in the proof of Lemma 2.4.1 and using the Comparison Principle for energy solutions (Lemma 2.1.7), we can build a sequence $\{u_n\}_{n\in\mathbb{N}}$ of energy solutions of the approximated problems (2.4.2), so that

$$u_0 \leqslant u_1 \leqslant ... \leqslant u_n \leqslant ... \leqslant \bar{u} \quad \text{ in } \mathbb{R}^N \times (0,T).$$

Hence, by the Monotone Convergence Theorem we can define $u := \lim_{n\to\infty} u_n \leqslant \bar{u}$. Moreover, applying the energy formulation of u_n,

$$\begin{aligned}
\|u_n\|^2_{L^2(0,T;H_0^s(\Omega))} &= \int_0^T \|u_n(\cdot,t)\|^2_{H_0^s(\Omega)}\,dt = \int_0^T \int_Q \frac{(u_n(x,t)-u_n(y,t))^2}{|x-y|^{N+2s}}\,dx\,dy\,dt \\
&= \frac{2}{a_{N,s}}\left\{\int_0^T\int_\Omega \left(\lambda\frac{u_n^2}{|x|^{2s}} + u_n^{p+1} + T_n(f)u_n\right)dx\,dt - \int_0^T\int_\Omega (u_n)_t u_n\,dx\,dt\right\} \\
&= \frac{2}{a_{N,s}}\left\{\int_0^T\int_\Omega \left(\lambda\frac{u_n^2}{|x|^{2s}} + u_n^{p+1} + T_n(f)u_n\right)dx\,dt - \frac{1}{2}\int_\Omega u_n(x,T)^2\,dx \right. \\
&\quad \left. +\frac{1}{2}\int_\Omega u_n(x,0)^2\,dx\right\} \\
&\leqslant \frac{2}{a_{N,s}}\left\{\int_0^T\int_\Omega \left(\lambda\frac{\bar{u}^2}{|x|^{2s}} + \bar{u}^{p+1} + f\bar{u}\right)dx\,dt + \frac{1}{2}\int_\Omega \bar{u}(x,0)^2\,dx\right\} \\
&\leqslant C.
\end{aligned}$$

Thus, up to a subsequence, we know that $u_n \rightharpoonup u$ in $L^2(0,T; H_0^s(\Omega))$. Likewise, for a fixed $0 \leqslant t \leqslant T$,

$$\begin{aligned}
\|(u_n)_t(\cdot,t)\|_{H^{-s}(\Omega)} &= \sup_{\|\varphi\|_{H_0^s(\Omega)}\leqslant 1}\left|\int_\Omega (u_n)_t(x,t)\varphi(x)\,dx\right| \\
&\leqslant \sup_{\|\varphi\|_{H_0^s(\Omega)}\leqslant 1}\left|\int_Q \frac{(u_n(x,t)-u_n(y,t))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy\right| \\
&\quad + \sup_{\|\varphi\|_{H_0^s(\Omega)}\leqslant 1}\left\{\int_\Omega u_n^p\varphi\,dx + \lambda\int_\Omega \frac{u_n\varphi}{|x|^{2s}}\,dx + \int_\Omega T_n(f)\varphi\right\} \\
&\leqslant \sup_{\|\varphi\|_{H_0^s(\Omega)}\leqslant 1}\left\{\|u_n(\cdot,t)\|_{H_0^s(\Omega)}\|\varphi\|_{H_0^s(\Omega)} + \int_\Omega \bar{u}^p\varphi\,dx + \lambda\int_\Omega\frac{\bar{u}\varphi}{|x|^{2s}}\,dx + \int_\Omega f\varphi\,dx\right\} \\
&\leqslant C\left(\|u_n(\cdot,t)\|_{H_0^s(\Omega)} + 1 + \|f\|_{L^2(\Omega)}\right).
\end{aligned}$$

Hence,

$$\int_0^T \|(u_n)_t\|^2_{H^{-s}(\Omega)}\,dt \leqslant C\left(\|u_n\|^2_{L^2(0,T;H_0^s(\Omega))}+1+\|f\|^2_{L^2(0,T;L^2(\Omega))}\right) \leqslant C,$$

and therefore, up to a subsequence, $(u_n)_t \rightharpoonup u_t$ in $L^2(0,T;H^{-s}(\Omega))$, and we can pass to the limit to conclude that u is a finite energy solution to (2.4.1).

□

2.4.1 Nonexistence results for $p > p(\lambda, s)$. Instantaneous and complete blow up.

We consider first the interval of powers $p > p(\lambda, s)$. In this case, analogously to the results exposed in Section 1.5 of Chapter 1 for the elliptic case, the goal here is not only proving a non existence result, but also a blow up phenomenon in problem (2.4.1).

Theorem 2.4.3. *Let $0 < \lambda \leqslant \Lambda_{N,s}$. If $p > p(\lambda, s)$, then problem* (2.4.1) *has no positive weak supersolution. In the case where $f \equiv 0$, the unique nonnegative supersolution is $u \equiv 0$.*

Proof. Without loss of generality, we can assume $f \in L^\infty(\Omega \times (0,T))$. We argue by contradiction. Assume that $\tilde{u}$ is a positive weak supersolution of (2.4.1). That is,

$$\tilde{u}_t + (-\Delta)^s\tilde{u} - \lambda\frac{\tilde{u}}{|x|^2} \geqslant 0 \text{ in } \Omega\times(0,T) \text{ in the weak sense.}$$

Since $\tilde{u}$ is also a weak supersolution in any $B_R(0)\times(T_1,T_2) \subset\subset \Omega\times(0,T)$, then by Lemma 2.4.1, the problem

$$\text{(2.4.5)}\qquad \begin{cases} u_t + (-\Delta)^s u = \lambda\dfrac{u}{|x|^{2s}} + u^p + f & \text{in } B_R(0)\times(T_1,T_2),\\ u(x,t) > 0 & \text{in } B_R(0)\times(T_1,T_2),\\ u(x,t) = 0 & \text{in } (\mathbb{R}^N\setminus B_R(0))\times[T_1,T_2),\\ u(x,T_1) = \tilde{u}(x,T_1) & \text{if } x\in B_R(0), \end{cases}$$

has a solution u obtained by approximation of the truncated problems in $B_R(0)\times(T_1,T_2)$. In particular $u := \lim u_n$, with $u_n \leqslant u$, and $u_n \in L^\infty(B_R(0)$ being the energy solution to (2.4.2) in $B_R(0)\times(T_1,T_2)$.

On the other hand, consider the energy solution v to the problem

$$\text{(2.4.6)}\qquad \begin{cases} v_t + (-\Delta)^s v = \lambda\dfrac{v}{|x|^{2s}} + f & \text{in } B_R(0)\times(T_1,T_2),\\ v(x,t) > 0 & \text{in } B_R(0)\times(T_1,T_2),\\ v(x,t) = 0 & \text{in } (\mathbb{R}^N\setminus B_R(0))\times[T_1,T_2),\\ v(x,T_1) = \tilde{u}(x,T_1) & \text{if } x\in B_R(0). \end{cases}$$

By Theorem 2.2.15, it can be seen that for any cylinder $B_r(0)\times(t_1,t_2)$, with $0<r<r_1<R$ and $0<T_1<t_1<t_2<T_2\leqslant T$, there exists a constant $C=C(N,r_1,t_1,t_2)$ such that $v\geqslant C|x|^{-\gamma}$ in $B_r(0)\times(t_1,t_2)$, with γ defined in (1.2.3). Moreover, by Lemma 2.1.9,

$$u\geqslant v\geqslant C|x|^{-\gamma} \text{ in } B_r(0)\times(t_1,t_2),$$

and then, for r small enough, $u>1$ in $B_r(0)\times(t_1,t_2)$. In particular, since u is in $L^1(\Omega\times(0,T))$, $\log(u)\in L^p(B_r(0)\times(t_1,t_2))$ for all $p\geqslant 1$. By a suitable scaling, we can assume that the cylinder is $B_r(0)\times(0,\tau)$.

Let $\phi\in\mathcal{C}_0^\infty(B_r(0))$. Then using $\dfrac{|\phi|^2}{u_n}$ as a test function in the approximated problems (2.4.2) and applying the Picone (Theorem 1.1.6) and Sobolev (Theorem 0.0.4) inequalities,

$$\begin{aligned}\int_0^\tau\int_{B_r(0)}\frac{u_{n-1}^p}{u_n}\phi^2\,dx\,dt &\leqslant \int_0^\tau\int_{B_r(0)}\frac{|\phi|^2}{u_n}u_{nt}dxdt+\int_0^\tau\int_{B_r(0)}(-\Delta)^s u_n\frac{|\phi|^2}{u_n}dx\,dt\\ &\leqslant \int_{B_r(0)}|\log u_n(x,\tau)|\phi^2\,dx+C(N,s,\tau)\|\phi\|^2_{H_0^s(\Omega)}.\end{aligned}$$

Therefore, passing to the limit as n tends to infinity, and using that $u\geqslant C|x|^{-\gamma}$ in $B_r(0)\times(0,\tau)$, we obtain

$$\begin{aligned}\int_{B_r(0)}|\log u(x,\tau)|\phi^2\,dx+C(N,s,\tau)\|\phi\|^2_{H_0^s(\Omega)} &\geqslant \int_0^\tau\int_{B_r(0)}u^{p-1}\phi^2\,dx\,dt\\ &\geqslant C\int_0^\tau\int_{B_r(0)}\frac{\phi^2}{|x|^{(p-1)\gamma}}dxdt.\end{aligned}$$

Using Hölder and Sobolev inequalities, it follows that

$$\begin{aligned}\int_{B_r(0)}|\log(u(x,\tau))||\phi|^2dx &\leqslant \left(\int_{B_r(0)}|\phi|^{2_s^*}\,dx\right)^{\frac{2}{2_s^*}}\left(\int_{B_r(0)}|\log u(x,\tau)|^{\frac{N}{2s}}\,dx\right)^{\frac{2s}{N}}\\ &\leqslant C\left(\int_{B_r(0)}|\log u(x,\tau)|^{\frac{N}{2s}}\,dx\right)^{\frac{2s}{N}}S\|\phi\|^2_{H_0^s(\Omega)},\end{aligned}$$

and therefore,

$$\|\phi\|^2_{H_0^s(\Omega)}\geqslant C\int_{B_r(0)}\frac{\phi^2}{|x|^{(p-1)\gamma}}dx.$$

Since $p>p(\lambda,s)$, $(p-1)\left(\dfrac{N-2s}{2}-\alpha\right)>2s$, and we obtain a contradiction with the Hardy inequality performing the same argument as in the proof of Theorem 1.5.1. $\square$

As we advanced, the previous nonexistence result is very strong in the sense that a complete and instantaneous blow up phenomenon occurs. That is, if $\{u_n\}_{n\in\mathbb{N}}$ are the solutions to the approximated problems (2.4.2), then $u_n(x,t)\to+\infty$ as $n\to+\infty$, where (x,t) is an arbitrary point in $\Omega\times(0,T)$. This is the parabolic version of Definition 1.5.2 in Chapter 1.

Theorem 2.4.4. *Let u_n be a solution to the problem (2.4.2) with $p > p(\lambda, s)$. Then $u_n(x_0, t_0) \to +\infty$ when $n \to +\infty$, for every $(x_0, t_0) \in \Omega \times (0, T)$.*

Proof. Without loss of generality, we can assume that $\lambda \leqslant \Lambda_N$. The existence of a positive solution to problem (2.4.2) is clear and, due to the Comparison Principle, we know that $u_n \leqslant u_{n+1}$ for all $n \in \mathbb{N}$.

Suppose by contradiction that there exists $(x_0, t_0) \in \Omega \times (0, T)$ such that

$$u_n(x_0, t_0) \to C_0 < +\infty \quad \text{as } n \to +\infty.$$

By the parabolic Harnack inequality (Theorem 2.1.11), there exists $s_0 > 0$ and a positive constant $C = C(N, s_0, t_0, \beta)$ such that

$$\iint_{R_0^-} u_n(x,t)\, dx\, dt \leqslant C \operatorname*{ess\,inf}_{R_0^+} u_n \leqslant C,$$

where $R_0^- := B_{s_0}(x_0) \times (t_0 - \frac{3}{4}\beta, t_0 - \frac{1}{4}\beta)$ and $R_0^+ := B_{s_0}(x_0) \times (t_0 + \frac{1}{4}\beta, t_0 + \frac{3}{4}\beta)$.

Without loss of generality, we can suppose $x_0 = 0$. Otherwise, we can find a finite sequence of points $\{x_i\}_{i=0}^M$, ending with $x_M = 0$, and a sequence of radius $\{s_i\}_{i=0}^M$ such that $B_{s_i}(x_i) \subset \Omega$, $B_{s_i}(x_i) \cap B_{s_{i+1}}(x_{i+1}) \neq \emptyset$, for all $i = 0, \ldots, M$ and, by the Hanarck inequality,

$$\iint_{R_i^-} u_n(x,t)\, dx\, dt \leqslant C \operatorname*{ess\,inf}_{R_i^+} u_n,$$

where $R_i^- = B_{s_i}(x_i) \times (t_i - \frac{3}{4}\beta, t_i - \frac{1}{4}\beta)$ and $R_i^+ = B_{s_i}(x_i) \times (t_i + \frac{1}{4}\beta, t_i + \frac{3}{4}\beta)$, $t_i \in (0, T)$ and β is small enough so that $t_i - \frac{3}{4}\beta > 0$ and $t_i + \frac{3}{4}\beta < T$ for all $i = 0, \ldots, M$. Let us choose now $t_i = t_{i-1} - \beta$ for $i = 1, \ldots, M$. Note that in this case

$$(t_i + \frac{1}{4}\beta, t_i + \frac{3}{4}\beta) = (t_{i-1} - \frac{3}{4}\beta, t_{i-1} - \frac{1}{4}\beta),$$

and in particular, $R_i^+ \cap R_{i-1}^- \neq \emptyset$. Thus,

$$\begin{aligned}
\iint_{R_M^-} u_n(x,t)\, dx\, dt &\leqslant \operatorname*{ess\,inf}_{R_M^+} u_n(x,t) \leqslant \operatorname*{ess\,inf}_{R_M^+ \cap R_{M-1}^-} u_n(x,t) \\
&\leqslant \frac{1}{|R_M^+ \cap R_{M-1}^-|} \iint_{R_M^+ \cap R_{M-1}^-} u_n(x,t)\, dx\, dt \\
&\leqslant \frac{1}{|R_M^+ \cap R_{M-1}^-|} \iint_{R_{M-1}^-} u_n(x,t)\, dx\, dt \\
&\leqslant \ldots \leqslant C \iint_{R_0^-} u_n(x,t)\, dx\, dt \leqslant \tilde{C}.
\end{aligned}$$

Therefore, supposing $x_0 = 0$, by the Monotone Convergence Theorem there exists $u \geqslant 0$ such that $u_n \nearrow u$ in $L^1(R_0^-)$. By simplicity, we suppose that this cylinder is $B_r(0) \times (t_1, t_2)$.

Let now φ be the solution to the problem

$$(2.4.7)\qquad \begin{cases} -\varphi_t + (-\Delta)^s\varphi = F & \text{in } \Omega\times(0,T),\\ \varphi(x,t) > 0 & \text{in } \Omega\times(0,T),\\ \varphi(x,t) = 0 & \text{in } (\mathbb{R}^N\setminus\Omega)\times[0,T),\\ \varphi(x,T) = 0 & \text{in } \Omega, \end{cases}$$

with $F \in \mathcal{C}_0^\infty(B_r(0)\times[t_1,t_2])$, and $0 \lneq F \leqslant 1$. Note that, due to the regularity of the right hand sides of problems (2.4.2) and (2.4.7), both u_n and φ are in the energy space, and thus both can be used as test functions in the energy formulation of the problems.

Indeed, considering first u_n as test function in (2.4.7) and then, after integrating by parts, φ in (2.4.2), and defining $\eta := \inf_{B_r(0)\times(t_1,t_2)}\varphi(x,t)$, we have

$$\begin{aligned} C &\geqslant \int_{t_1}^{t_2}\int_{B_r(0)} u_n(x,t)\,dx\,dt \geqslant \int_{t_1}^{t_2}\int_{B_r(0)} u_nF\,dx\,dt\\ &\geqslant \lambda\int_0^T\int_\Omega \frac{u_{n-1}}{|x|^{2s}+\frac{1}{n}}\varphi\,dx\,dt + \int_0^T\int_\Omega u_{n-1}^p\varphi\,dx\,dt + \int_0^T\int_\Omega T_n(f)\varphi\,dx\,dt\\ &\geqslant \lambda\eta\int_{t_1}^{t_2}\int_{B_r(0)} \frac{u_{n-1}}{|x|^{2s}+\frac{1}{n}}\,dx\,dt + \eta\int_{t_1}^{t_2}\int_{B_r(0)} u_{n-1}^p\,dx\,dt + \eta\int_{t_1}^{t_2}\int_{B_r(0)} T_n(f)\,dx\,dt. \end{aligned}$$

By the Monotone Convergence Theorem,

$$\frac{u_{n-1}}{|x|^{2s}} + u_{n-1}^p + T_n(f) \nearrow \frac{u}{|x|^{2s}} + u^p + f \text{ in } L^1(B_r(0)\times(t_1,t_2)).$$

Thus it follows that u is a weak supersolution to (2.4.1) in $B_r(0)\times(t_1,t_2)$, a contradiction with Theorem 2.4.3. ☐

2.4.2 Existence results for $1 < p < p(\lambda, s)$.

The goal now is to consider the complementary interval of powers, $1 < p < p(\lambda, s)$, and to prove that under some suitable hypotheses on f and u_0, problem (2.4.1) has a positive solution or, equivalently, the optimality of $p(\lambda, s)$. We will consider here the case $f \equiv 0$. For the case $f \not\equiv 0$, see Remark 2.4.6 at the end of this subsection.

First of all, notice that if $0 < \lambda \leqslant \Lambda_{N,s}$ and $1 < p < p(\lambda, s)$, the stationary problem

$$(2.4.8)\qquad (-\Delta)^s u = \lambda\frac{u}{|x|^{2s}} + u^p \text{ in } \Omega,\quad u > 0 \text{ in } \Omega,\quad u = 0 \text{ in } \mathbb{R}^N\setminus\Omega,$$

has a positive supersolution w, depending on the following cases:

(A) $0 < \lambda < \Lambda_{N,s}$: In Proposition 1.3.2 in Chapter 1, we found a positive solution to the problem

$$(2.4.9)\qquad (-\Delta)^s u = \lambda\frac{u}{|x|^{2s}} + u^p + \mu u^q \text{ in } \Omega,\quad u > 0 \text{ in } \Omega,\quad u = 0 \text{ in } \mathbb{R}^N\setminus\Omega,$$

for μ small enough, $0 < q < 1$ and $1 < p < p(\lambda, s)$. In particular, this solution is a supersolution of (2.4.8). Note that for $1 < p \leqslant 2_s^* - 1$ this supersolution is in the energy space, and for $2_s^* - 1 < p < p(\lambda, s)$, it is a weak positive supersolution.

(B) If $\lambda = \Lambda_{N,s}$, then $p(\lambda, s) = 2_s^* - 1$. Thus, instead of $H_0^s(\Omega)$, we consider the Hilbert space $H(\Omega)$ defined in (2.1.5). Since $H(\Omega)$ is compactly embedded in $L^p(\Omega)$ for all $1 \leqslant p < 2_s^*$, classical variational methods in the space $H(\Omega)$ allow us to prove the existence of a positive solution w to the stationary problem (2.4.8).

Theorem 2.4.5. *Assume that* $0 < \lambda \leqslant \Lambda_{N,s}$ *and* $1 < p < p(\lambda, s)$. *Suppose that* $u_0(x) \leqslant \overline{w}$, *where* $\overline{w}$ *is a supersolution to the stationary problem*

$$(-\Delta)^s w = \lambda \frac{w}{|x|^{2s}} + w^p \text{ in } \Omega, \quad w(x) > 0 \text{ in } \Omega, \quad w(x) = 0 \text{ on } \mathbb{R}^N \setminus \Omega.$$

Then for all $T > 0$, *the problem*

$$\left\{\begin{array}{ll} u_t + (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + u^p & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\ u(x,0) = u_0(x) & \text{if } x \in \Omega, \end{array}\right. \tag{2.4.10}$$

has a global positive solution. If $\overline{w}$ *is a weak supersolution, the solution will be also weak, and likewise, if* $\overline{w}$ *is an energy supersolution, problem* (2.4.10) *will have an energy solution.*

Proof. Since $\overline{w}(x) \geqslant u_0(x)$ for all $x \in \Omega$, then $\overline{w}$ is a positive supersolution to problem (2.4.10). Hence, we conclude just by applying Lemma 2.4.1, whether $\overline{w}$ is a weak supersolution, or Lemma 2.4.2, if $\overline{w}$ is an energy supersolution. ☐

Remark 2.4.6. *In the presence of a source term* $f \gneqq 0$, *if* $f(x,t) \leqslant \frac{c_0(t)}{|x|^{2s}}$ *with* $c_0(t)$ *bounded and sufficiently small, the computation above allows us to prove the existence of a supersolution. Then the existence of a minimal solution to problem* (2.4.1) *follows for all* $p < p(\lambda, s)$.

Chapter 3

An elliptic problem with a singular nonlinearity

In this chapter we study the existence and regularity of solutions of the following problem,

$$
\begin{cases}
(-\Delta)^s u = F(x,u) := \dfrac{f(x)}{u^\alpha} + M u^p & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases} \tag{3.0.1}
$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N > 2s$, $M \in \{0,1\}$, $s \in (0,1)$, $\alpha > 0$, $p > 1$, and f is a nonnegative function that belongs to some $L^m(\Omega)$, $m \geqslant 1$.

Such elliptic problems with a singular nonlinearity have a large history in the local case that is, when $s = 1$. Indeed, the seminal paper by Crandall, Rabinowitz and Tartar [73] is the starting point of a large literature, see for instance [27, 28, 42, 39, 59, 74, 115, 121, 129, 132, 133, 168, 181]. Notice that, although a singular term appears in the right hand side, if we consider the variational formulation of the problem, for $0 < \alpha < 1$ the singular term turns into a positive power, and one can somehow understand this case as a kind of concave-convex nonlinearity. See for instance the papers [16, 40, 50, 109, 112] and their corresponding references.

The strategy will be to study at the begining the case $M = 0$, in order to use solutions of this problem as subsolutions of the convex case. Indeed, the case $M = 0$ is strongly inspired by the semilinear elliptic problem

$$
\begin{cases}
-\Delta u = \dfrac{f(x)}{u^\alpha} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{in } \partial\Omega,
\end{cases} \tag{3.0.2}
$$

where $N \geqslant 2$, $\alpha > 0$ is a real number and f is a nonnegative function that belongs to some Lebesgue space. This singular problem appears by considering problems with a *convection term* via a change of variable, i.e., it is related to singular problems with a term depending on $|\nabla v|^2$, where v is the new variable. See for instance the references [115, 181].

In [42], the authors study existence and regularity results of the solutions to the problem (3.0.2), depending on the value of α and on the summability of f. Our aim will be to prove, using similar techniques as in the local case, this kind of results for the nonlocal framework. In particular we will work by approximation, that is, analyzing the problems obtained truncating the singular term $u^{-\alpha}$ and the datum f, so that the first one becomes non singular at the origin and the second one belongs to $L^\infty(\Omega)$ (see [42, 39] in the local setting).

For the case $M = 1$, the motivation arises from the following semilinear problem, whose nonlinearity combines a singular term and a convex one

$$\begin{cases} -\Delta u = \dfrac{\mu}{u^\alpha} + u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where α, p and μ are positive numbers (see, among other papers, [27, 28, 39, 59, 74, 121, 181] for an extensive analysis of this kind of problems in the case $s = 1$). The multiplicity behavior in this case (both for $s = 1$ and $s \in (0, 1)$) is essentially the same as in *concave-convex* type problems and, since we already studied this case in Chapter 1, we will focus here on finding the first solution for the optimal range of μ. More precisely, we will prove the existence of a solution for $\alpha > 0$ and $p > 1$, up to some threshold on μ, by means of an approximation method.

Furthermore, to complement the results obtained in Chapter 1, in the last section we will study the problem

$$\begin{cases} (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + \dfrac{h(x)}{u^\alpha} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < \lambda < \Lambda_{N,s}$, $\alpha > 0$ and h satisfies some summability condition that will be specified in every case. Notice that in that previous chapter, we studied the problem with the Hardy potential and concave-convex powers, that is, powers between 0 and a threshold found in (1.2.6). In this sense, this problem studies the complementary interval, that is, the negative powers, and therefore extends the results in Chapter 1 to every power in $\mathbb{R}$. It is worth to point out that, as far as we know, the results contained in this section are new also for the local case.

The results in Sections 3.1 - 3.3 of this chapter can be found in [33]. Section 3.4 will appear in [3].

3.1 Preliminaries and functional setting.

First of all, we need to precise the sense of solutions that we will handle here. In particular, if we consider the general problem

$$(D)\begin{cases}(-\Delta)^s u = F(x,u) & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

as we already did in Chapter 1 and Chapter 2, we will distinguish two types of solutions, attending to the regularity of F and u. Indeed,

Definition 3.1.1. We say that $u\in H_0^s(\Omega)$ is a positive *energy* supersolution (respectively subsolution) of problem (D) if $F\in L_{loc}^{(2_s^*)'}(\Omega)$ and

$$\frac{a_{N,s}}{2}\iint_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \geqslant (\leqslant) \int_\Omega F(x,u)\varphi(x)\,dx, \tag{3.1.1}$$

for every nonnegative $\varphi\in H_0^s(\Omega)$ with compact support contained in Ω.

If u is a supersolution and a subsolution of (D), we say that it is a positive energy solution.

Analogously, when we have less regularity on u, we will make use of a weaker notion of solution. Define first the set

(3.1.2)
$$\widetilde{\mathcal{T}} := \{\phi:\mathbb{R}^N\to\mathbb{R} \text{ measurable s.t. } (-\Delta)^s\phi=\varphi\in L^\infty(\Omega),\ \phi=0 \text{ on } \mathbb{R}^N\setminus\widetilde{\Omega},\ \widetilde{\Omega}\subset\subset\Omega\}.$$

Notice that $\widetilde{\mathcal{T}}\subseteq\mathcal{T}$, where $\mathcal{T}$ was defined in (1.1.2).

Definition 3.1.2. We say that $u\in L^1(\Omega)$ is a positive *weak* supersolution (respectively subsolution) of problem (D) if $F\in L^1_{loc}(\Omega)$, $u=0$ in $\mathbb{R}^N\setminus\Omega$ and

$$\int_\Omega u(-\Delta)^s\phi\,dx \geqslant (\leqslant) \int_\Omega F(x,u)\phi(x)\,dx, \tag{3.1.3}$$

for every nonnegative $\phi\in\tilde{\mathcal{T}}$.

We say that u is a positive weak solution of problem (D) if it is at the same time a supersolution and a subsolution of such a problem.

Notice that, due to the singular term in (3.0.1), differently from the cases of Chapter 1 and Chapter 2, only by asking $u\in H_0^s(\Omega)$ we cannot expect $F(x,u)$ to belong to $L^{(2_s^*)'}(\Omega)$ in the first case, or $F(x,u)\in L^1(\Omega)$ if $u\in L^1(\Omega)$ in the second one. Thus, the right hand side is not well defined if we test in the natural spaces $H_0^s(\Omega)$ and $\mathcal{T}$ respectively. To deal with this difficulty we restrict our test sets to the functions with compact support.

Remark 3.1.3. *If $F \in L^{(2_s^*)'}(\Omega)$ in (D), we can extend Definition 3.1.1, saying that* (3.1.1) *holds for every $\varphi \in H_0^s(\Omega)$. This will be the case of the approximated problems,*

$$\begin{cases} (-\Delta)^s u_n = \lambda \dfrac{\min(f(x), n)}{\left(u_n + \frac{1}{n}\right)^\gamma} + M u_n^p & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

that we will use to build solutions of problem (3.0.1)*. Notice that here the first term in the right hand side is no longer singular (it is bounded indeed), so we do not need to restrict our test set to functions compactly supported in Ω.*

3.2 The singular problem. Solvability and regularity.

To start analyzing the problem (3.0.1), in this section we will focus in the case $M = 0$, that is, when we only have the singular term in the right hand side. In particular, we consider the problem

$$(D_\alpha) \begin{cases} (-\Delta)^s u = \dfrac{f(x)}{u^\alpha} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\alpha > 0$ and the summability of f will be conveniently specified later. Notice that studying this case we will understand the actual singular behavior of problem (D), that is the main difference with respect to the previous chapters. Moreover, the results from this section will be crucial to face the case $M = 1$, since the solutions constructed here will work as subsolutions for the problem with the convex term.

3.2.1 Approximated problems.

In order to study the solvability of problem (D_α), we will analyze the associated approximated problems. Indeed, suppose $f \in L^1(\Omega)$, $f \geqslant 0$, and consider the problem

$$(D_{n,\alpha}) \begin{cases} (-\Delta)^s u_n = \dfrac{f_n}{\left(u_n + \frac{1}{n}\right)^\alpha} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $f_n := T_n(f)$ for every $n \in \mathbb{N}$.

Lemma 3.2.1. *Problem $(D_{n,\alpha})$ has a nonnegative solution $u_n \in H_0^s(\Omega) \cap L^\infty(\Omega)$.*

Proof. Fix $n \in \mathbb{N}$. Let $v \in H_0^s(\Omega)$, and define $w := T(v)$ as the unique energy solution to the problem

$$\begin{cases} (-\Delta)^s w = \dfrac{f_n}{\left(v^+ + \frac{1}{n}\right)^\alpha} & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.2.1}$$

Notice that the existence and uniqueness of solution to this problem is given by the Lax-Milgram Theorem, since the right hand side belongs to the dual space $H^{-s}(\Omega)$. Testing now in (3.2.1) with w, we get

$$\begin{aligned} \frac{a_{N,s}}{2} \iint_Q \frac{(w(x)-w(y))^2}{|x-y|^{N+2s}}\,dx\,dy = \int_\Omega \frac{f_n w}{\left(v^+ + \frac{1}{n}\right)^\alpha}\,dx &\leqslant n^{\alpha+1}\|w\|_{L^1(\Omega)} \\ &\leqslant C(N,s,\Omega) n^{\alpha+1}\|w\|_{L^{2_s^*}(\Omega)}, \end{aligned}$$

and thus, by the Sobolev embedding given in Theorem 0.0.4, it follows that

$$\|w\|_{H_0^s(\Omega)} \leqslant C n^{\alpha+1}, \tag{3.2.2}$$

with $C = C(N,s,\Omega)$ independent of v, so that the ball of radius $Cn^{\alpha+1}$ is invariant under T in $H_0^s(\Omega)$. In order to apply the Schauder's Fixed Point Theorem over T to guarantee the existence of a solution of $(D_{n,\alpha})$, apart from the invariance, we need to check the continuity and compactness of T as an operator from $H_0^s(\Omega)$ to $H_0^s(\Omega)$.
We prove first the continuity. In order to do this, we want to check that, if we denote $w_k := T(v_k)$ and $w := T(v)$, then

$$\lim_{k\to\infty} \|w_k - w\|_{H_0^s(\Omega)} = 0 \text{ whenever } \lim_{k\to\infty} \|v_k - v\|_{H_0^s(\Omega)} = 0. \tag{3.2.3}$$

Notice that from the convergence of v_k in $H_0^s(\Omega)$, by Theorem 0.0.4 we obtain

$$\begin{aligned} &v_k \to v \text{ in } L^{2_s^*}(\Omega), \\ &v_k \to v \text{ a.e. in } \Omega. \end{aligned} \tag{3.2.4}$$

In fact, let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence in $H_0^s(\Omega)$ converging to $v \in H_0^s(\Omega)$. Thus we get

(3.2.5)

$$\begin{aligned} \frac{a_{N,s}}{2}\|w_k - w\|_{H_0^s(\Omega)}^2 \leqslant & \int_\Omega \left(\frac{f_n}{\left(v_k^+ + \frac{1}{n}\right)^\alpha} - \frac{f_n}{\left(v^+ + \frac{1}{n}\right)^\alpha} \right)(w_k - w)\,dx \\ \leqslant & \|w_k - w\|_{L^{2_s^*}(\Omega)} \left(\int_\Omega \left(\frac{f_n}{\left(v_k^+ + \frac{1}{n}\right)^\alpha} - \frac{f_n}{\left(v^+ + \frac{1}{n}\right)^\alpha} \right)^{(2_s^*)'} dx \right)^{1/(2_s^*)'}, \end{aligned}$$

where $(2_s^*)' = \frac{2N}{N+2s} < 2_s^*$. Thus, by Hölder's inequality and Theorem 0.0.4 again, we obtain

$$\|w_k - w\|_{H_0^s(\Omega)} \leqslant C(N,s,n,\Omega) \left(\int_\Omega \left(\frac{f_n}{\left(v_k^+ + \frac{1}{n}\right)^\alpha} - \frac{f_n}{\left(v^+ + \frac{1}{n}\right)^\alpha} \right)^{2_s^*} dx \right)^{1/2_s^*}.$$

Now we observe that both

$$\frac{f_n}{\left(v_k^+ + \frac{1}{n}\right)^\alpha} \leqslant n^{\alpha+1} \quad \text{and} \quad \frac{f_n}{\left(v^+ + \frac{1}{n}\right)^\alpha} \leqslant n^{\alpha+1},$$

and therefore, by the Dominated Convergence Theorem and (3.2.4), we conclude that

$$\|w_k - w\|_{H_0^s(\Omega)} \to 0 \text{ as } k \to \infty,$$

and hence, T is continuous from $H_0^s(\Omega)$ to $H_0^s(\Omega)$.

To see that T is compact, we take a sequence $\{v_k\}_{k\in\mathbb{N}}$ such that $\|v_k\|_{H_0^s(\Omega)} \leqslant C$. Therefore, by Rellich-Kondrachov Theorem (see Theorem 0.0.5) we conclude that, up to a subsequence,

$$\begin{aligned} &v_k \rightharpoonup v \text{ in } H_0^s(\Omega), \\ &v_k \to v \text{ in } L^r(\Omega),\ 1 \leqslant r < 2_s^*. \end{aligned} \tag{3.2.6}$$

Furthermore, since T is continuous,

$$\|T(v_k)\|_{H_0^s(\Omega)} \leqslant C,$$

with C a positive constant independent of k, and hence,

$$\begin{aligned} &T(v_k) \rightharpoonup \tilde{w} \text{ in } H_0^s(\Omega), \\ &T(v_k) \to \tilde{w} \text{ in } L^r(\Omega),\ 1 \leqslant r < 2_s^*. \end{aligned} \tag{3.2.7}$$

Because of the continuity of T, necessarily $\tilde{w} = T(v)$. Thus, proceeding as in (3.2.5) one can reach

$$\begin{aligned} &\frac{a_{N,s}}{2} \|T(v_k) - T(v)\|_{H_0^s(\Omega)}^2 \\ &\qquad \leqslant \|T(v_k) - T(v)\|_{L^2(\Omega)} \left(\int_\Omega \left(\frac{f_n}{\left(v_k^+ + \frac{1}{n}\right)^\alpha} - \frac{f_n}{\left(v^+ + \frac{1}{n}\right)^\alpha} \right)^2 dx \right)^{1/2}, \end{aligned}$$

and by (3.2.7) we conclude

$$\lim_{k\to\infty} \|T(v_k) - T(v)\|_{H_0^s(\Omega)} = 0,$$

and therefore T is compact from $H_0^s(\Omega)$ to $H_0^s(\Omega)$.

Given these conditions on T, Schauder's Fixed Point Theorem provides the existence of $u_n \in H^s_0(\Omega)$ such that $u_n = T(u_n)$, i.e. u_n solves

$$\begin{cases} (-\Delta)^s u_n = \dfrac{f_n}{\left(u_n^+ + \frac{1}{n}\right)^\alpha} & \text{in } \Omega, \\ \qquad u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.2.8}$$

By Lemma 1.1.4, $u_n \geqslant 0$, and hence u_n solves $(D_{n,\alpha})$. Finally, since the right hand side of $(D_{n,\alpha})$ belongs to $L^\infty(\Omega)$, by [134, Corollary 3] we also get that $u_n \in L^\infty(\Omega)$. □

Moreover, we can prove the following result, that tightly follows the local argument performed by L. Boccardo and L. Orsina in [42].

Lemma 3.2.2. *Let u_n be a solution to $(D_{n,\alpha})$. Thus, $\{u_n\}_{n\in\mathbb{N}}$ is an increasing sequence, $u_n > 0$ in Ω, and for every set $\tilde{\Omega} \subset\subset \Omega$ there exists a positive constant $c_{\tilde{\Omega}}$, independent of n, such that*

$$u_n(x) \geqslant c_{\tilde{\Omega}} > 0, \quad \textit{for every } x \in \tilde{\Omega} \textit{ and every } n \in \mathbb{N}. \tag{3.2.9}$$

Proof. Consider the problems satisfied by u_n and u_{n+1}. Substracting them we get

$$\begin{aligned} (-\Delta)^s(u_n - u_{n+1}) &= \frac{f_n}{\left(u_n + \frac{1}{n}\right)^\alpha} - \frac{f_{n+1}}{\left(u_{n+1} + \frac{1}{n+1}\right)^\alpha} \\ &\leqslant \frac{f_{n+1}}{\left(u_n + \frac{1}{n+1}\right)^\alpha} - \frac{f_{n+1}}{\left(u_{n+1} + \frac{1}{n+1}\right)^\alpha} \\ &= f_{n+1} \frac{\left(u_{n+1} + \frac{1}{n+1}\right)^\alpha - \left(u_n + \frac{1}{n+1}\right)^\alpha}{\left(u_n + \frac{1}{n+1}\right)^\alpha \left(u_{n+1} + \frac{1}{n+1}\right)^\alpha}. \end{aligned}$$

Now we choose $(u_n - u_{n+1})_+$ as test function here. Since

$$\left[\left(u_{n+1} + \frac{1}{n+1}\right)^\alpha - \left(u_n + \frac{1}{n+1}\right)^\alpha\right](u_n - u_{n+1})_+ \leqslant 0,$$

proceeding as in the proof of Lemma 1.1.4, we conclude that $u_n \leqslant u_{n+1}$.

On the other hand, from Lemma 3.2.1 we know that u_1 belongs to $L^\infty(\Omega)$, that is, there exists $C > 0$ such that $\|u_1\|_{L^\infty(\Omega)} \leqslant C$, and thus,

$$(-\Delta)^s u_1 = \frac{f_1}{(u_1 + 1)^\alpha} \geqslant \frac{f_1}{(\|u_1\|_{L^\infty(\Omega)} + 1)^\alpha} \geqslant \frac{f_1}{(C+1)^\alpha}.$$

Moreover, by [150, Proposition 1.1], $u_1 \in \mathcal{C}^s(\mathbb{R}^N)$ and hence, since $\dfrac{f_1}{(C+1)^\alpha}$ is not identically zero, a further use of the Strong Maximum Principle ([160, Proposition 2.17]) gives us that $u_1 > 0$ in Ω and hence, by the monotonicity of $\{u_n\}_{n\in\mathbb{N}}$, (3.2.9) holds for every $n \geqslant 1$. □

Corollary 3.2.3. *The solution u_n to problem $(D_{n,\alpha})$ is unique.*

Proof. Let us consider $v_n \neq u_n$ a solution of $(D_{n,\alpha})$. Taking $(u_n - v_n)^+$ ($(v_n - u_n)^+$ resp.) as a test function in $(D_{n,\alpha})$, we conclude $v_n \leqslant u_n$ ($u_n \leqslant v_n$ resp.), and the uniqueness follows. □

Now, the goal is passing to the limit in the sequence $\{u_n\}_{n\in\mathbb{N}}$ to achieve a solution of (D_α). With this purpose, we must distinguish three cases, attending to the value of the power α.

3.2.2 Case $\alpha \leqslant 1$.

We prove in this case that, if $f \in L^{(2_s^*)'}(\Omega)$, we can build an energy solution of (D_α).

Lemma 3.2.4. *Let u_n be the solution of problem $(D_{n,\alpha})$, and $f \gneq 0$. Then,*

- *if $\alpha = 1$ and $f \in L^1(\Omega)$, or*
- *if $\alpha < 1$ and f belongs to $L^m(\Omega)$ with $m = \frac{2N}{N+2s+\alpha(N-2s)} = \left(\frac{2_s^*}{1-\alpha}\right)' > 1$,*

u_n is uniformly bounded in $H_0^s(\Omega)$.

Proof. Let us first consider the case $\alpha = 1$. Taking u_n as a test function in $(D_{n,\alpha})$, just by noticing that $\dfrac{u_n}{u_n + \frac{1}{n}} \leqslant 1$, one gets,

$$\frac{a_{N,s}}{2}\|u_n\|^2_{H_0^s(\Omega)} = \int_\Omega \frac{f_n u_n}{u_n + \frac{1}{n}}\,dx \leqslant \int_\Omega f\,dx < +\infty,$$

that is, $\|u_n\|_{H_0^s(\Omega)} \leqslant C$, with C independent of n.

In the case $\alpha < 1$ taking again u_n as a test function in $(D_{n,\alpha})$, by Theorem 0.0.4 and Hölder's inequality, we get

$$\begin{aligned}\frac{a_{N,s}}{2}\|u_n\|^2_{H_0^s(\Omega)} &\leqslant \int_\Omega f u_n^{1-\alpha}\,dx\\ &\leqslant \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{2_s^*}\,dx\right)^{\frac{1}{m'}}\\ &\leqslant C(N,s)\|f\|_{L^m(\Omega)}\|u_n\|^{\frac{2_s^*}{m'}}_{H_0^s(\Omega)}.\end{aligned}$$

Since $2 > \frac{2_s^*}{m'}$, we get a uniform estimate of u_n in the space $H_0^s(\Omega)$. □

Theorem 3.2.5. *Let $f \in L^{(2_s^*)'}(\Omega)$, $f \gneq 0$, and $\alpha \leqslant 1$. Then there exists an energy solution $u \in H_0^s(\Omega)$ of problem (D_α).*

Proof. First of all, notice that

$$(2_s^*)' = \frac{2N}{N+2s} \geqslant \frac{2N}{N+2s+\alpha(N-2s)} = \left(\frac{2_s^*}{1-\alpha}\right)' > 1.$$

Therefore, for both cases $\alpha = 1$ and $\alpha < 1$, by Lemma 3.2.4 the sequence $\{u_n\}_{n\in\mathbb{N}}$ of solutions to $(D_{n,\alpha})$ is uniformly bounded in $H_0^s(\Omega)$, and thus

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } H_0^s(\Omega), \\ u_n &\nearrow u \text{ a.e. in } \Omega. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \iint_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy = \iint_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy,$$

for every $\varphi \in H_0^s(\Omega)$. Moreover, if $\operatorname{supp}(\varphi) = \omega \subset\subset \Omega$, by (3.2.9) and using that f belongs to $L^{(2_s^*)'}(\Omega)$, we have

$$0 \leqslant \left| \frac{f_n\varphi}{\left(u_n+\frac{1}{n}\right)^\alpha} \right| \leqslant \frac{|f||\varphi|}{c_\omega^\alpha} \in L^1(\Omega).$$

Therefore, by the Dominated Convergence Theorem we get

$$\lim_{n\to\infty} \int_\Omega \frac{f_n\varphi}{\left(u_n+\frac{1}{n}\right)^\alpha}\,dx = \int_\Omega \frac{f\varphi}{u^\alpha}\,dx,$$

and we conclude that u is a positive energy solution of (D_α). ☐

3.2.3 Case $\alpha > 1$.

We prove in this case the existence of a weak solution to (D_α).

Lemma 3.2.6. *Let $f \in L^1(\Omega)$, $f \gneq 0$ and let u_n be the solution of problem $(D_{n,\alpha})$ for $\alpha > 1$. Then, $u_n^{\frac{\alpha+1}{2}}$ is uniformly bounded in $H_0^s(\Omega)$.*

Proof. Let $T > 1$. We consider, for $\beta > 1$, the convex function

$$\Phi_\beta(r) := \begin{cases} r^\beta & \text{if } 0 \leqslant r < T, \\ \beta T^{\beta-1} r - (\beta-1)T^\beta & \text{if } r \geqslant T. \end{cases} \tag{3.2.10}$$

Let us take $\beta := \frac{\alpha+1}{2} > 1$ and we call $\Phi(r) := \Phi_{\frac{\alpha+1}{2}}(r)$. Since $\Phi(r)$ is a Lipschitz function (with constant $L_{\Phi(r)} = \frac{\alpha+1}{2}T^{\frac{\alpha-1}{2}}$), then $\Phi(u_n)$ and $\Phi(u_n)\Phi'(u_n)$ belong to $H_0^s(\Omega)$ (see [134, Proposition 3]).

Using [134, Proposition 2.4], we have

$$(-\Delta)^s\Phi(u_n) \leqslant \Phi'(u_n)(-\Delta)^s u_n, \tag{3.2.11}$$

where this inequality is understood in the energy sense. Therefore

$$\begin{aligned}\int_{\mathbb{R}^N}(-\Delta)^{s/2}\Phi(u_n)(-\Delta)^{s/2}\Phi(u_n)\,dx &\leqslant \int_{\mathbb{R}^N}(-\Delta)^{s/2}\left(\Phi'(u_n)\Phi(u_n)\right)(-\Delta)^{s/2}u_n\,dx \\ &= \int_\Omega \frac{f_n}{\left(u_n+\frac{1}{n}\right)^\alpha}\Phi'(u_n)\Phi(u_n)\,dx.\end{aligned} \tag{3.2.12}$$

Since $\Phi'(u_n)\Phi(u_n) \leqslant \frac{\alpha+1}{2}u_n^\alpha$, from (3.2.12) it follows that

$$\frac{a_{N,s}}{2}\|\Phi(u_n)\|^2_{H_0^s(\Omega)} \leqslant \frac{(\alpha+1)}{2}\|f\|_{L^1(\Omega)} \leqslant C,$$

where $C>0$ is independent of n. Letting $T\to+\infty$ we conclude. □

Theorem 3.2.7. *Let $f\in L^1(\Omega)$, $f\gneq 0$, and $\alpha>1$. Then problem (D_α) admits a weak solution u. Moreover, $u^{\frac{\alpha+1}{2}}\in H_0^s(\Omega)$.*

Proof. Consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ of solutions to the problem $(D_{n,\alpha})$. By Lemma 3.2.2 we know that this sequence is increasing, and thus we can define $u := \lim_{n\to\infty} u_n$. Hence, due to the weak lower semicontinuity of the norm, by Lemma 3.2.6 we obtain

$$\|u^{\frac{\alpha+1}{2}}\|_{H_0^s(\Omega)} \leqslant \liminf_{n\to\infty}\|u_n^{\frac{\alpha+1}{2}}\|_{H_0^s(\Omega)} \leqslant C,$$

with C a positive constant independent of n, i.e., $u^{\frac{\alpha+1}{2}}\in H_0^s(\Omega)$. Moreover, by the Sobolev embedding this implies $u^{\frac{\alpha+1}{2}}\in L^{2_s^*}(\Omega)$ and, since $\frac{\alpha+1}{2}2_s^*>1$, in particular $u\in L^1(\Omega)$. Thus, for every $\phi\in\mathcal{T}$,

$$\lim_{n\to\infty}\int_\Omega u_n(-\Delta)^s\phi\,dx = \int_\Omega u(-\Delta)^s\phi\,dx < +\infty.$$

Moreover, if $\phi\in\widetilde{\mathcal{T}}$, then $\operatorname{supp}(\phi) =: \omega\subset\subset\Omega$, and by Lemma 3.2.2 we have

$$0\leqslant\left|\frac{f_n\phi}{(u_n+\frac{1}{n})^\alpha}\right|\leqslant\frac{|f||\phi|}{c_\omega^\alpha}\in L^1(\Omega).$$

Therefore, by the Dominated Convergence Theorem we can pass to the limit in the right hand side of the weak formulation of $(D_{n,\alpha})$, to conclude that $u\in L^1(\Omega)$ is a weak solution of problem (D_α). □

3.2.4 Summability of the solutions according to the summability of the data.

We begin by considerating $\alpha\leqslant 1$.

Proposition 3.2.8. *Let $f\in L^{(2_s^*)'}(\Omega)$, $0<\alpha\leqslant 1$, and let u be the solution of (D_α) provided by Theorem 3.2.5. Then $u\in L^{(\alpha+1)2_s^*}(\Omega)$.*

Proof. Following the classical method, explained in the nonlocal setting in [134, Theorem 3.10], we consider $\Phi_\beta(r)$, the convex function defined in (3.2.10), with $\beta := \alpha + 1$, as a test function in $(D_{n,\alpha})$. Proceeding as in Lemma 3.2.6, since $\Phi'_\beta(u_n)\Phi_\beta(u_n) \leqslant \beta u_n^{\beta+1}$, we get that

(3.2.13)
$$\begin{aligned}\int_{\mathbb{R}^N} (-\Delta)^{s/2}\Phi_\beta(u_n)(-\Delta)^{s/2}\Phi_\beta(u_n)\,dx &\leqslant \int_{\mathbb{R}^N} (-\Delta)^{s/2}\left(\Phi'_\beta(u_n)\Phi_\beta(u_n)\right)(-\Delta)^{s/2}u_n\,dx \\ &\leqslant \int_\Omega \frac{f}{u_n}\Phi'_\beta(u_n)\Phi_\beta(u_n)\,dx \leqslant \beta\int_\Omega f u_n^\beta\,dx.\end{aligned}$$

The integral in the left hand side of (3.2.13) can be estimated, by Theorem 0.0.4, in the following way,

(3.2.14)
$$\begin{aligned}\int_{\mathbb{R}^N}\left|(-\Delta)^{s/2}\Phi_\beta(u_n)\right|^2 dx = \frac{a_{N,s}}{2}\|\Phi_\beta(u_n)\|^2_{H_0^s(\Omega)} &\geqslant C(N,s)\|\Phi_\beta(u_n)\|^2_{L^{2_s^*}(\Omega)} \\ &= C(N,s)\left[\int_{\{u_n<T\}} u_n^{\beta 2_s^*}\,dx + \int_{\{u_n\geqslant T\}}(\beta T^{\beta-1}u_n - (\beta-1)T^\beta)^{2_s^*}\,dx\right]^{\frac{2}{2_s^*}} \\ &\geqslant C(N,s)\left[\int_{\{u_n<T\}} u_n^{\beta 2_s^*}\,dx + \int_{\{u_n\geqslant T\}} T^{\beta 2_s^*}\,dx\right]^{\frac{2}{2_s^*}} \\ &\geqslant C(N,s)\left[\int_{\{u_n<T\}} u_n^{\beta 2_s^*}\,dx + \operatorname{meas}\{u_n\geqslant T\}\right]^{\frac{2}{2_s^*}},\end{aligned}$$

where in the last step we have used the fact $T > 1$. Moreover, since $u_n \in H_0^s(\Omega)$,

$$\lim_{T\to+\infty}(\operatorname{meas}\{u_n \geqslant T\}) = 0.$$

Thus, letting $T \to +\infty$, from (3.2.13) and (3.2.14) we find that

$$\|u_n^\beta\|^2_{L^{2_s^*}(\Omega)} \leqslant C(N,s)\int_\Omega f u_n^\beta\,dx \leqslant C(N,s)\|f\|_{L^{(2_s^*)'}(\Omega)}\|u_n^\beta\|_{L^{2_s^*}(\Omega)}.$$

Hence, by Fatou's Lemma,

$$\|u^\beta\|_{L^{2_s^*}(\Omega)} \leqslant \liminf_{n\to\infty}\|u_n^\beta\|_{L^{2_s^*}(\Omega)} \leqslant C,$$

and we conclude that u belongs to $L^{(\alpha+1)2_s^*}(\Omega)$. □

Remark 3.2.9. *Observe that the exponent of summability $(\alpha+1)2_s^*$ coincides when $s = 1$ with the one given in [5, Lemmas 3.3 and 5.5] in the local case.*

The summability of the solution obtained in the previous proposition could be improved requiring more regularity to the function f. In order to prove this result, we will adapt to the nonlocal framework the ideas given in [28, Lemma 1], to obtain the following:

Lemma 3.2.10. *Let $\mu > 0$ and $\alpha > 0$. If $w \in H_0^s(\Omega)$, $w > 0$, satisfies*

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} w (-\Delta)^{s/2} u \, dx \leqslant \mu \int_\Omega \frac{\phi}{w^\alpha} \qquad \text{for every } \phi \in H_0^s(\Omega), \tag{3.2.15}$$

then there exists a constant $C > 0$, independent of w, such that

$$\|w\|_{L^\infty(\Omega)} \leqslant C \mu^{\frac{1}{\alpha+1}}.$$

Proof. Let us first consider the case $\mu = 1$. For $k \geqslant 1$, we test in (3.2.15) with $G_k(w)$ (see (1.1.8) for the definition of G_k). By [134, Proposition 2.7, ii)], we get that

$$\frac{a_{N,s}}{2} \|G_k(w)\|^2_{H_0^s(\Omega)} \leqslant \int_{A_k} \frac{G_k(w)}{w^\alpha} \, dx \leqslant \int_{A_k} G_k(w) \, dx \leqslant \|G_k(w)\|_{L^{2_s^*}(\Omega)} |A_k|^{\frac{N+2s}{2N}},$$

where

$$A_k := \{x \in \Omega : \; w(x) > k\}.$$

Thus, by Theorem 0.0.4,

$$\|G_k(w)\|_{L^{2_s^*}(\Omega)} \leqslant C(N,s) |A_k|^{\frac{N+2s}{2N}}. \tag{3.2.16}$$

Furthermore, taking $z > k$, there holds

$$\|G_k(w)\|_{L^{2_s^*}(\Omega)} \geqslant \|G_k(w)\|_{L^{2_s^*}(A_z)} \geqslant (z-k)|A_z|^{1/2_s^*}, \tag{3.2.17}$$

and plugging this into (3.2.16), we obtain

$$|A_z| \leqslant \frac{C}{(z-k)^{2_s^*}} |A_k|^{\frac{N+2s}{N-2s}}.$$

Hence, by Lemma 2.2.24, we conclude that there exists a positive constant $C = C(N,s)$ such that $\|w\|_{L^\infty(\Omega)} \leqslant C$.
The general case $\mu > 0$, easily follows using the linearity of the fractional Laplacian. In fact, if we consider the modified function

$$\widetilde{w} := \left(\frac{1}{\mu}\right)^{\frac{1}{\alpha+1}} w \in H_0^s(\Omega),$$

for every $\varphi \in H_0^s(\Omega)$ we get that

$$\begin{aligned} \int_{\mathbb{R}^N} (-\Delta)^{s/2} \widetilde{w} (-\Delta)^{s/2} \varphi \, dx &= \left(\frac{1}{\mu}\right)^{\frac{1}{\alpha+1}} \int_{\mathbb{R}^N} (-\Delta)^{s/2} w (-\Delta)^{s/2} \varphi \, dx \\ &\leqslant \left(\frac{1}{\mu}\right)^{\frac{1}{\alpha+1}} \int_\Omega \frac{\mu\varphi}{w^\alpha} \, dx = \int_\Omega \frac{\varphi}{\widetilde{w}^\alpha} \, dx. \end{aligned}$$

That is, $\widetilde{w}$ is an energy subsolution of (3.2.15) with $\mu = 1$. Thus, because of the previous computations, there exists $C > 0$ such that $\|\widetilde{w}\|_{L^\infty(\Omega)} \leqslant C$. Therefore,

$$\|w\|_{L^\infty(\Omega)} \leqslant C \mu^{\frac{1}{\alpha+1}}.$$

□

Analogously, we can state the next result.

Proposition 3.2.11. *If u is the solution of (D_α) provided by Theorem 3.2.5 or by Theorem 3.2.7 and $f \in L^m(\Omega)$ with $m > \frac{N}{2s}$, then $u \in L^\infty(\Omega)$.*

The result follows again as a consequence of Stampacchia's iteration result for the solutions u_n to the truncated problems (D_n, α), so we omit the proof. In this case, one proves the existence of a constant $C = C(N, s, \|f\|_{L^m(\Omega)}) > 0$ such that

$$\|u_n\|_{L^\infty(\Omega)} \leqslant C,$$

and recalling that $u := \lim_{n\to\infty} u_n$ we conclude the result.

3.3 Solvability of the singular problem with a convex term.

In this section, we study the influence of a convex term of the form u^p, with $p > 1$, in the previous problem (D_α). Indeed, we will deal with the problem

$$(D_{\mu,\alpha,p}) \begin{cases} (-\Delta)^s u = \dfrac{\mu}{u^\alpha} + u^p & \text{in } \Omega, \\ \qquad u > 0 & \text{in } \Omega, \\ \qquad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

As we advanced at the begining of the Chapter, in the case $0 < \alpha < 1$, the singular term can be seen as a concave term, and thus one can expect a multiplicity result similar to the concave-convex problems, that we already analyzed in Chapter 1. Nevertheless, in this section we will focus on proving the existence of at least one solution for every $\alpha > 0$ and $p > 1$.

Theorem 3.3.1. *Assume $\alpha > 0$ and $p > 1$. Then, there exists $0 < \Upsilon < +\infty$ such that, for every $0 < \mu < \Upsilon$ there exists a positive solution u to the problem $(D_{\mu,\alpha,p})$ in the following sense:*

- *if $0 < \alpha \leqslant 1$, then $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$ is an energy solution;*
- *if $\alpha > 1$, then $u \in L^\infty(\Omega)$ is a weak solution, satisfying $u^{\frac{\alpha+1}{2}} \in H_0^s(\Omega)$.*

Moreover, $(D_{\mu,\alpha,p})$ does not have a positive solution whenever $\mu > \Upsilon$.

Proof. Step 1:
Consider the following approximated problems,

$$(D_{n,\mu,\alpha,p}) \begin{cases} (-\Delta)^s u_n = \dfrac{\mu}{\left(u_n^+ + \frac{1}{n}\right)^\alpha} + (u_n^+)^p & \text{in } \Omega, \\ \qquad u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

First, we will prove that there exists a solution of this problem, by applying the Sattinger method, and then we will try to pass to the limit to obtain a solution to $(D_{\mu,\alpha,p})$.

Step 2: Subsolution.
Let $\underline{u_n} \in H_0^s(\Omega) \cap L^\infty(\Omega)$ be the solution to the problem

$$\begin{cases} (-\Delta)^s \underline{u_n} = \dfrac{\mu}{\left(\underline{u_n}^+ + \frac{1}{n}\right)^\alpha} & \text{in } \Omega, \\ \underline{u_n} = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.3.1}$$

given by Lemma 3.2.1. By Lemma 1.1.4, $\underline{u_n} \geqslant 0$, and thus

$$\frac{\mu}{\left(\underline{u_n}^+ + \frac{1}{n}\right)^\alpha} \leqslant \frac{\mu}{\left(\underline{u_n} + \frac{1}{n}\right)^\alpha} + \underline{u_n}^p,$$

that is, $\underline{u_n}$ is a subsolution of $(D_{n,\mu,\alpha,p})$.
Step 3: Supersolution.
Let $t > \mu$, that will be chosen later, and let $\overline{u_n} \geqslant 0$ be the energy solution to the problem

$$\begin{cases} (-\Delta)^s \overline{u_n} = \dfrac{t}{\left(\overline{u_n}^+ + \frac{1}{n}\right)^\alpha} & \text{in } \Omega, \\ \overline{u_n} = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.3.2}$$

By Lemma 3.2.10, we know that there exists a constant $C_0 > 0$, independent of n and t, such that

$$\|\overline{u_n}\|_{L^\infty(\Omega)} \leqslant C_0 t^{\frac{1}{\alpha+1}}. \tag{3.3.3}$$

In order to see that $\overline{u_n}$ is actually a supersolution of $(D_{n,\mu,\alpha,p})$, we need to prove that we can choose t large enough so that

$$\frac{t}{\left(\overline{u_n} + \frac{1}{n}\right)^\alpha} \geqslant \frac{\mu}{\left(\overline{u_n} + \frac{1}{n}\right)^\alpha} + \overline{u_n}^p. \tag{3.3.4}$$

But this is equivalent to

$$t \geqslant \mu + \overline{u_n}^p \left(\overline{u_n} + \frac{1}{n}\right)^\alpha,$$

what, as a consequence of (3.3.3), in particular holds if

$$t \geqslant \mu + (C_0 t^{\frac{1}{\alpha+1}})^p \left(C_0 t^{\frac{1}{\alpha+1}} + \frac{1}{n}\right)^\alpha. \tag{3.3.5}$$

Notice first that, since $\frac{\alpha+p}{\alpha+1} > 1$, for μ small enough we can find some $t > 0$ such that

$$t \geqslant \mu + 2^\alpha C_0^{p+\alpha} t^{\frac{\alpha+p}{\alpha+1}}.$$

Hence, choosing

$$\frac{1}{n} \leqslant C_0 t^{\frac{1}{\alpha+1}},$$

(3.3.5) holds. Thus, we have proved the existence of $\Upsilon_0 > 0$ such that for $0 < \mu < \Upsilon_0$, we can build a supersolution $\overline{u_n}$ to the problem $(D_{n,\mu,\alpha,p})$.

Step 4: $\underline{u_n} \leqslant \overline{u_n}$.
We proceed as in the proof of Lemma 3.2.2, that is, we consider the problem satisfied by $\underline{u_n} - \overline{u_n}$ and we use $(\underline{u_n} - \overline{u_n})^+$ as test function. Therefore, since $\mu < t$, and $\left(r + \frac{1}{n}\right)^{-\alpha}$ is decreasing for $r > 0$, we get that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta)^{s/2}(\underline{u_n} - \overline{u_n})^+ (-\Delta)^{s/2}(\underline{u_n} - \overline{u_n})\, dx \\
&\qquad = \int_\Omega \left(\frac{\mu}{\left(\underline{u_n} + \frac{1}{n}\right)^\alpha} - \frac{t}{\left(\overline{u_n} + \frac{1}{n}\right)^\alpha} \right) (\underline{u_n} - \overline{u_n})^+\, dx \\
&\qquad = \int_\Omega \left(\frac{\mu}{\left(\underline{u_n} + \frac{1}{n}\right)^\alpha} - \frac{\mu}{\left(\overline{u_n} + \frac{1}{n}\right)^\alpha} \right) (\underline{u_n} - \overline{u_n})^+\, dx \\
&\qquad\quad + \int_\Omega \left(\frac{\mu}{\left(\overline{u_n} + \frac{1}{n}\right)^\alpha} - \frac{t}{\left(\overline{u_n} + \frac{1}{n}\right)^\alpha} \right) (\underline{u_n} - \overline{u_n})^+\, dx \\
&\qquad \leqslant 0.
\end{aligned}
$$

Reasoning as in the proof of Lemma 1.1.4, we conclude that $\underline{u_n} \leqslant \overline{u_n}$.

Step 5: Sattinger method.
Consider the increasing function

$$
g(r) := \frac{\mu}{\left(r + \frac{1}{n}\right)^\alpha} + r^p + n^{\alpha+1}\mu\alpha r, \quad r \in [0, +\infty),
$$

and let $u_{n,1} \in H_0^s(\Omega)$ be the solution to the problem

$$
\begin{cases}
(-\Delta)^s u_{n,1} + n^{\alpha+1}\mu\alpha u_{n,1} = g(\underline{u_n}) & \text{in } \Omega, \\
u_{n,1} > 0 & \text{in } \Omega, \\
u_{n,1} = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\tag{3.3.6}
$$

Notice that $u_{n,1}$ exists because, since $\underline{u_n} \in L^\infty(\Omega)$, the right hand side is linear and bounded, and the right hand side conforms a bounded coercive bilinear map. Thus, Lax-Milgram Theorem ensures the existence and uniqueness of $u_{n,1} \in H_0^s(\Omega)$. Moreover,

$$
(-\Delta)^s(\underline{u_n} - u_{n,1}) + n^{\alpha+1}\mu\alpha(\underline{u_n} - u_{n,1}) \leqslant g(\underline{u_n}) - g(\underline{u_n}) = 0,
$$

and testing again with $(\underline{u_n} - u_{n,1})^+$, we conclude $\underline{u_n} \leqslant u_{n,1}$.

Likewise, it can be proved that $u_{n,1} \leqslant \overline{u_n}$. Consider now for every $k \in \mathbb{N}$ the iterated problems

$$
\begin{cases}
(-\Delta)^s u_{n,k+1} + n^{\alpha+1}\mu\alpha u_{n,k+1} = g(u_{n,k}) & \text{in } \Omega, \\
u_{n,k+1} > 0 & \text{in } \Omega, \\
u_{n,k+1} = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\tag{3.3.7}
$$

Then, since g is increasing, the solutions of these problems satisfy

$$\underline{u_n} \leqslant u_{n,1} \leqslant \cdots \leqslant u_{n,k} \leqslant u_{n,k+1} \leqslant \overline{u_n}.$$

Thus, we can define the pointwise limit $u_n := \lim_{k\to\infty} u_{n,k}$. Moreover, using $u_{n,k+1}$ as a test function in (3.3.7), by Lemma 3.2.10, we get that

$$\begin{aligned}
\frac{a_{N,s}}{2}\|u_{n,k+1}\|^2_{H^s_0(\Omega)} &+ n^{\alpha+1}\mu\int_\Omega \alpha u^2_{n,k+1}\,dx \\
&= \mu\int_\Omega \frac{u_{n,k+1}}{\left(u_{n,k}+\frac{1}{n}\right)^\alpha}\,dx + \int_\Omega u^p_{n,k}u_{n,k+1}\,dx + n^{\alpha+1}\mu\int_\Omega \alpha u_{n,k}u_{n,k+1}\,dx \\
&\leqslant \mu\int_\Omega n^\alpha \overline{u_n}\,dx + \int_\Omega \overline{u_n}^{p+1}\,dx + n^{\alpha+1}\mu\int_\Omega \alpha\overline{u_n}^2\,dx \\
&\leqslant C_n,
\end{aligned}$$

where C_n is a constant dependent on n but independent of k. Hence, up to a subsequence, we can conclude that $u_{n,k} \rightharpoonup u_n$ in $H^s_0(\Omega)$. Thus, u_n is an energy solution of problem $(D_{n,\mu,\alpha,p})$.

Furthermore, by Lemma 3.2.2, since the sequence of subsolutions $\{\underline{u_n}\}_{n\in\mathbb{N}}$ is increasing with respect to n, there exists a constant $c_{\tilde\Omega} > 0$, independent of n, such that

$$u_n \geqslant \underline{u_n} \geqslant \underline{u_1} \geqslant c_{\tilde\Omega} > 0, \text{ for every } x \in \tilde\Omega. \tag{3.3.8}$$

Remark 3.3.2. *Note that, by construction, the solution u_n of problem $(D_{n,\mu,\alpha,p})$ is a minimal solution, that is, if $\widetilde{u_n}$ is another solution of $(D_{n,\mu,\alpha,p})$ then $u_n \leqslant \widetilde{u_n}$.*

Step 6: Regularity.
The purpose now is to pass to the limit in the sequence $\{u_n\}_{n\in\mathbb{N}}$ in order to get a solution of $(D_{\mu,\alpha,p})$. Consider first the case $\alpha \leqslant 1$. Using $u_n \in H^s_0(\Omega)$ as a test function in the problem $(D_{n,\mu,\alpha,p})$, one gets

$$\begin{aligned}
\frac{a_{N,s}}{2}\|u_n\|^2_{H^s_0(\Omega)} &= \mu\int_\Omega \frac{u_n}{\left(u_n+\frac{1}{n}\right)^\alpha}\,dx + \int_\Omega u_n^{p+1}\,dx \\
&\leqslant \mu\int_\Omega u_n^{1-\alpha}\,dx + \int_\Omega u_n^{p+1}\,dx \\
&\leqslant C_1\|\overline{u_n}\|^{1-\alpha}_{L^\infty(\Omega)} + C_2\|\overline{u_n}\|^{p+1}_{L^\infty(\Omega)} \\
&\leqslant C,
\end{aligned}$$

where, by Lemma 3.2.10, C is a constant independent of n. Therefore, u_n is uniformly bounded in $H^s_0(\Omega)$ and, up to a subsequence,

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } H^s_0(\Omega),\\
&u_n \to u \text{ in } L^r(\Omega),\ 1 \leqslant r < 2^*_s.
\end{aligned}$$

Moreover, for every $\varphi \in H^s_0(\Omega)$ with $\mathrm{supp}(\varphi) \subset \tilde\Omega \subset\subset \Omega$, by (3.3.8), we have that

$$0 \leqslant \frac{\mu\varphi}{\left(u_n+\frac{1}{n}\right)^\alpha} \leqslant \frac{\mu\varphi}{c^\alpha_{\tilde\Omega}} \in L^1(\Omega).$$

Thus, we can pass to the limit in the energy formulation of the approximated problems $(D_{n,\mu,\alpha,p})$ to conclude that u is an energy solution of $(D_{\mu,\alpha,p})$.

Consider now $\alpha > 1$. We proceed as in the proof of Lemma 3.2.6, considering the Lipschitz convex function $\Phi_{\frac{\alpha+1}{2}}(r)$, defined in (3.2.10). Therefore, by Lemma 3.2.10, we have that

$$\begin{aligned}
\frac{a_{N,s}}{2}\|\Phi_{\frac{\alpha+1}{2}}(u_n)\|^2_{H_0^s(\Omega)} &\leqslant \int_{\mathbb{R}^N}(-\Delta)^{s/2}\left(\Phi'_{\frac{\alpha+1}{2}}(u_n)\Phi_{\frac{\alpha+1}{2}}(u_n)\right)(-\Delta)^{s/2}u_n\,dx\\
&= \int_\Omega \frac{\mu}{(u_n+\frac{1}{n})^\alpha}\Phi'_{\frac{\alpha+1}{2}}(u_n)\Phi_{\frac{\alpha+1}{2}}(u_n)\,dx + \int_\Omega u_n^p\Phi'_{\frac{\alpha+1}{2}}(u_n)\Phi_{\frac{\alpha+1}{2}}(u_n)\,dx\\
&\leqslant \frac{(\alpha+1)\mu}{2}\int_\Omega \frac{u_n^\alpha}{\left(u_n+\frac{1}{n}\right)^\alpha}\,dx + \frac{\alpha+1}{2}\int_\Omega u_n^{p+\alpha}\,dx\\
&\leqslant \frac{(\alpha+1)\mu}{2}|\Omega| + \frac{\alpha+1}{2}\|u_n\|^{p+\alpha}_{L^\infty(\Omega)}|\Omega|\\
&\leqslant C,
\end{aligned}$$

where C is a positive constant independent of n. Letting $T \to +\infty$ in the definition of $\Phi_{\frac{\alpha+1}{2}}$, we conclude that $u_n^{\frac{\gamma+1}{2}}$ is uniformly bounded in $H_0^s(\Omega)$, and thus, by the Rellich-Kondrachov Theorem (Theorem 0.0.5), there exists $u^{\frac{\alpha+1}{2}} \in H_0^s(\Omega)$ such that

$$\begin{aligned}
&u_n^{\frac{\alpha+1}{2}} \to u^{\frac{\alpha+1}{2}} \text{ in } L^r(\Omega),\ \forall\, 1 \leqslant r < 2_s^*,\\
&u_n^{\frac{\alpha+1}{2}} \to u^{\frac{\alpha+1}{2}} \text{ a.e. in } \Omega.
\end{aligned}$$

In particular, from here we deduce that $u_n \to u$ a.e. in Ω, and from Lemma 3.2.10, that $u \in L^\infty(\Omega)$. Hence, by this convergence and (3.3.8), we can pass to the limit in the weak formulation of $(D_{n,\mu,\alpha,p})$ to get

$$\int_{\mathbb{R}^N} u(-\Delta)^s\phi\,dx = \mu\int_\Omega \frac{\phi}{u^\alpha}\,dx + \int_\Omega u^p\phi\,dx,$$

for every $\phi \in \tilde{\mathcal{T}}$, that is, to conclude that $u \in L^\infty(\Omega)$ is a weak solution of $(D_{n,\mu,\alpha,p})$.

Step 7: $\Upsilon < +\infty$.
Let us define

$$\Upsilon := \sup\{\mu > 0 \text{ such that problem } (D_{\mu,\alpha,p}) \text{ has a solution}\}. \tag{3.3.9}$$

Following the ideas of [39, Remark 2.2] we will prove nonexistence for large μ, that is, $\Upsilon < +\infty$. Let $\Omega' \subset\subset \Omega$ and consider the eigenvalue problem

$$\begin{cases} (-\Delta)^s\varphi_1 = \mu_1\varphi_1 & \text{in } \Omega',\\ \varphi_1 > 0 & \text{in } \Omega',\\ \varphi_1 = 0 & \text{in } \mathbb{R}^N\setminus\Omega'. \end{cases} \tag{3.3.10}$$

By [157, Proposition 4] we know that $0 \leqslant \varphi_1 \in H_0^s(\Omega') \cap L^\infty(\Omega')$. Moreover, by [150, Proposition 1.1], $\varphi_1 \in \mathcal{C}^s(\mathbb{R}^N) \cap \mathcal{C}^{2s+\beta}(\Omega')$, $\beta > 0$, and (3.3.10) is satisfied pointwise. Furthermore, $\varphi_1 \in H_0^s(\Omega)$ and

$$(-\Delta)^s \varphi_1(x) \leqslant 0 = \mu_1 \varphi_1(x), \; x \in \Omega \setminus \Omega'. \tag{3.3.11}$$

Let u be a solution of $(D_{\mu,\alpha,p})$. Then, by testing in $(D_{\mu,\alpha,p})$ with φ_1 and applying (3.3.11) and Young's inequality, one gets

$$\mu \int_\Omega \frac{\varphi_1}{u^\alpha}\, dx + \int_\Omega u^p \varphi_1 \, dx \leqslant \mu_1 \int_\Omega u\varphi_1 \, dx \leqslant \frac{1}{p} \int_\Omega u^p \varphi_1 \, dx + \frac{\mu_1^{p'}}{p'} \int_\Omega \varphi_1 \, dx,$$

that is,

$$\int_\Omega \left(\frac{\mu}{u^\alpha} + \frac{p-1}{p} u^p - \frac{\mu_1^{p'}}{p'} \right) \varphi_1 \, dx \leqslant 0. \tag{3.3.12}$$

But it can be seen that

$$C := \left(\frac{p-1}{\alpha} \right)^{\frac{\alpha}{p+\alpha}} + \frac{p-1}{p} \left(\frac{\alpha}{p-1} \right)^{\frac{p}{p+\alpha}} > 0,$$

satisfies

$$\frac{\mu}{u^\alpha} + \frac{p-1}{p} u^p \geqslant C \mu^{\frac{p}{p+\alpha}},$$

and hence, (3.3.12) implies that

$$\int_\Omega \left(C \mu^{\frac{p}{p+\alpha}} - \frac{\mu_1^{p'}}{p'} \right) \varphi_1 \, dx \leqslant 0,$$

impossible for μ large enough.

Step 8: There exists at least a solution of $(D_{\mu,\alpha,p})$ for every $0 < \mu < \Upsilon$.
In Step 5 and Step 6 we have proved the existence of solutions for $0 < \mu < \Upsilon_0$, where Υ_0 was small enough so that we could construct the supersolution of Step 3. The purpose now is to prove that indeed we can find a solution for every $0 < \mu < \Upsilon$, where Υ was defined in (3.3.9).

Take $0 < \mu < \Upsilon$. Given the definition of Υ, we can find $\bar{\mu}$ as close as we want to Υ so that problem $(D_{\bar{\mu},\alpha,p})$ has a solution $u_{\bar{\mu}}$. In particular, taking $\mu < \bar{\mu} < \Upsilon$, it is easy to check that $u_{\bar{\mu}}$ is a supersolution of $(D_{n,\mu,\alpha,p})$. Proceeding as in Step 4, one can prove that $\underline{u_n} \leqslant u_{\bar{\mu}}$, where $\underline{u_n}$ is the subsolution of $(D_{n,\mu,\alpha,p})$ constructed at Step 2. Therefore, repeating Step 5 and Step 6 with the new supersolution $u_{\bar{\mu}}$, the result follows. □

Remark 3.3.3. *Notice that, in the case $0 < \alpha < 1$, problem $(D_{\mu,\alpha,p})$ has variational structure. Indeed, it can be checked that critical points of the functional $\mathcal{G} : H_0^s(\Omega) \to \mathbb{R}$ defined as*

$$\mathcal{G}(u) := \frac{a_{N,s}}{4} \|u\|^2_{H_0^s(\Omega)} - \frac{\mu}{1-\alpha} \int_\Omega u_+^{1-\alpha} \, dx - \frac{1}{p+1} \int_\Omega u_+^{p+1} \, dx.$$

correspond to solutions of such problem.

Furthermore, it has the form of a concave-convex problem, and it easily follows that, proceeding as we did in Chapter 1, for μ small enough this functional satisfies the Palais-Smale condition and the Mountain Pass Geometry (see Proposition 1.4.3 and Proposition 1.4.1). Hence, there exists some value Υ_0 such that, if $\mu \in (0, \Upsilon_0)$, problem $(D_{\mu,\alpha,p})$ has at least two positive energy solutions.

3.4 Interplay with the Hardy potential.

Finally, to conclude this chapter, we are going to analyze the relation between the previous sections and Chapter 1. More precisely, there we studied the problem

$$\begin{cases} (-\Delta)^s u - \lambda \dfrac{u}{|x|^{2s}} = u^p + \mu u^q & \text{in } \Omega, \\ \qquad\qquad u > 0 & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $0 < q < 1$, $0 < \lambda < \Lambda_{N,s}$, and $1 < p < p(\lambda, s)$, where $p(\lambda, s)$ was the supercritical threshold detailed in (1.2.6). In particular, we proved existence in this range of p, and non existence whether $p > p(\lambda, s)$. Thus, a natural question arising from there it is precisely what happens if we consider the complementay interval of powers, that is, $p < 0$. But translating this question to the notation we have been following in this chapter, this is nothing but asking about the behavior of the problem

$$\begin{cases} (-\Delta)^s u = \lambda \dfrac{u}{|x|^{2s}} + \dfrac{h(x)}{u^\alpha} & \text{in } \Omega, \\ \qquad u > 0 & \text{in } \Omega, \\ \qquad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.4.1}$$

with h a nonnegative function with suitable summability conditions. That is, we want to study if the results in Section 3.2 hold when we add the Hardy potential. To find solutions to this problem, we will proceed again by approximation. In particular, let us consider the approximated problem

$$\begin{cases} (-\Delta)^s u_n = \lambda \dfrac{u_n}{|x|^{2s}} + \dfrac{h_n}{\left(u_n + \frac{1}{n}\right)^\alpha} & \text{in } \Omega, \\ \qquad u_n > 0 & \text{in } \Omega, \\ \qquad u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.4.2}$$

where $h_n := T_n(h)$. Thus,

Lemma 3.4.1. *Let $0 \leqslant \lambda < \Lambda_{N,s}$. Then, problem (3.4.2) has a positive energy solution $u_n \in H_0^s(\Omega)$. Moreover, the sequence $\{u_n\}_{n\in\mathbb{N}}$ of solutions is increasing and, for every set $\tilde{\Omega} \subset\subset \Omega$, there exists a constant $c_{\tilde{\Omega}} > 0$, independent of n, such that*

$$u_n(x) \geqslant c_{\tilde{\Omega}} > 0, \text{ for every } x \in \tilde{\Omega} \text{ and every } n \in \mathbb{N}.$$

Proof. The existence follows reproducing the proof of Lemma 3.2.1 (that is, as a consequence of Lax-Milgram theorem and a fixed point argument), starting with the linear problem

$$\begin{cases} (-\Delta)^s w - \lambda \dfrac{w}{|x|^{2s}} = \dfrac{T_n(h)}{\left(v^+ + \frac{1}{n}\right)^\alpha} & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.4.3}$$

Indeed, since $0 \leqslant \lambda < \Lambda_{N,s}$, by the Hardy inequality (see Theorem 0.0.6), the operator in the left hand side of this problem is coercive in $H_0^s(\Omega)$, and thus the Lax-Milgram theorem provides an energy solution. The fixed point argument works in the same way, just by appropriately applying the Hardy inequality.

Likewise, as a consequence of this inequality, we can easily adapt the proof of Lemma 3.2.2 to conclude the second part of the statement. □

We analyze first the conditions needed to obtain energy solutions to problem (3.4.1).

Proposition 3.4.2. *Let $0 \leqslant \lambda < \Lambda_{N,s}$ and $0 < \alpha \leqslant 1$. Then,*

- *if $\alpha = 1$ and $h \in L^1(\Omega)$, $h \gneqq 0$, or*
- *if $0 < \alpha < 1$ and $h \in L^{(2_s^*)'}(\Omega)$, $h \gneqq 0$,*

problem (3.4.1) *has a positive energy solution.*

Proof. Let u_n be the solution to (3.4.2) provided by Lemma 3.4.1. Notice that, as a consequence of the Hardy inequality, if we test with u_n in (3.4.2), we obtain

$$\frac{a_{N,s}}{2}\left(1 - \frac{\lambda}{\Lambda_{N,s}}\right) \|u_n\|^2_{H_0^s(\Omega)} \leqslant \int_\Omega h u_n^{1-\alpha}\, dx,$$

and proceeding as in the proof of Lemma 3.2.4 we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $H_0^s(\Omega)$. Passing to the limit as in Theorem 3.2.5 we get the existence of energy solution. □

On the other hand, analogously to the results previously obtained, when we have a larger power in the singularity, we will need to weaken the notion of solutions, so we will obtain a weak solution instead of an energy one. To prove this, we will make use of the following auxiliar result.

Lemma 3.4.3. *Let $s_1, s_2 \geqslant 0$ and $\alpha > 0$. Then*

$$(s_1 - s_2)(s_1^\alpha - s_2^\alpha) \geqslant \frac{4\alpha}{(\alpha+1)^2}\left(s_1^{\frac{\alpha+1}{2}} - s_2^{\frac{\alpha+1}{2}}\right)^2. \tag{3.4.4}$$

Proof. If $s_1 = 0$ or $s_2 = 0$, the inequality trivially follows.

Thus, without loss of generality, we can suppose $s_1 > s_2 > 0$. Thus setting $x := \dfrac{s_2}{s_1}$, (3.4.4) is equivalent to

$$(1-x)(1-x^{\alpha}) \geqslant \frac{4\alpha}{(\alpha+1)^2}(1-x^{\frac{\alpha+1}{2}})^2 \text{ for all } x \in (0,1). \tag{3.4.5}$$

We set

$$g(x) := (1-x)(1-x^{\alpha})(\alpha+1)^2 - 4\alpha(1-x^{\frac{\alpha+1}{2}})^2,$$

and then we just have to see that $g(x) \geqslant 0$ for all $x \in (0,1)$. It can be checked that g may also be written as

$$g(x) = (\alpha-1)^2(1-x^{\frac{\alpha+1}{2}})^2 - (\alpha+1)^2(x^{\frac{1}{2}} - x^{\frac{\alpha}{2}})^2.$$

Assume first $\alpha \geqslant 1$. Thus, we claim that

$$(\alpha-1)(1-x^{\frac{\alpha+1}{2}}) \geqslant (\alpha+1)(x^{\frac{1}{2}} - x^{\frac{\alpha}{2}}).$$

Indeed, let us define

$$g_1(x) := (\alpha-1)(1-x^{\frac{\alpha+1}{2}}) - (\alpha+1)(x^{\frac{1}{2}} - x^{\frac{\alpha}{2}}).$$

Then,

$$g_1'(x) = \frac{(\alpha+1)}{2}\Big(-(\alpha-1)x^{\frac{\alpha-1}{2}} - x^{-\frac{1}{2}} + \alpha x^{\frac{\alpha-2}{2}}\Big).$$

Using Young's inequality with exponents $p := \frac{\alpha}{\alpha-1}$ and $q := \alpha$, we obtain

$$x^{\frac{\alpha}{2}-1} = x^{\frac{(\alpha-1)^2}{2\alpha}} x^{-\frac{1}{2\alpha}} \leqslant \frac{\alpha-1}{\alpha} x^{\frac{\alpha-1}{2}} + \frac{1}{\alpha} x^{-\frac{1}{2}}.$$

Thus $g_1'(x) \leqslant 0$ and hence $g_1(x) \geqslant g_1(1) = 0$. Therefore $g(x) \geqslant 0$ and (3.4.5) holds.

Consider now the case $\alpha < 1$. Hence, we prove the result if we show

$$(1-\alpha)(1-x^{\frac{\alpha+1}{2}}) \geqslant (\alpha+1)(x^{\frac{\alpha}{2}} - x^{\frac{1}{2}}).$$

Defining

$$g_2(x) := (1-\alpha)(1-x^{\frac{\alpha+1}{2}}) - (\alpha+1)(x^{\frac{\alpha}{2}} - x^{\frac{1}{2}}),$$

and using again Young's inequality we obtain $g_2'(x) \leqslant 0$ for all $x \in (0,1)$. Thus,

$$g_2(x) \geqslant g_2(1) = 0,$$

and we conclude. □

Now, our goal is to prove the existence of a weak solution in the case $\alpha > 1$, since it makes no sense to search for an energy solution. Nevertheless, although our solution will not belong to $H_0^s(\Omega)$, it will be an energy solution in *some local sense*. More precisely, we will define the local H^s space as

$$H_{loc}^s(\Omega) := \{u \in L_{loc}^2(\Omega) : \forall\, \Omega_1 \subset\subset \Omega, \iint_{\Omega_1\times\Omega_1} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\Omega_1} u^2\,dx < +\infty\}.$$

Thus,

Theorem 3.4.4. *Assume $\alpha > 1$ and $\lambda < \Lambda_{N,s}$. Then for all $h \in L^1(\Omega)$, $h \gneq 0$, problem (3.4.1) has a positive weak solution $u \in L^1(\Omega)$, satisfying $u \in H_{loc}^s(\Omega)$ with $G_k(u) \in H_0^s(\Omega)$ and $T_k(u)^{\frac{\alpha+1}{2}} \in H_0^s(\Omega)$. Furthermore, if*

$$\left(\frac{4\alpha}{(\alpha+1)^2} - \frac{\lambda}{\Lambda_{N,s}}\right) > 0,$$

then $u^{\frac{\alpha+1}{2}} \in H_0^s(\Omega)$.

Proof. Consider the energy solution u_n to the approximated problem (3.4.2), and let us take $G_k(u_n) \in H_0^s(\Omega)$, defined in (1.1.8), as a test function in that problem. By [134, Proposition 3],

$$\frac{a_{N,s}}{2}\iint_Q \frac{|G_k(u_n(x)) - G_k(u_n(y))|^2}{|x-y|^{N+2s}}\,dx\,dy - \lambda\int_\Omega \frac{u_n G_k(u_n)}{|x|^{2s}}\,dx \leqslant \int_\Omega \frac{h_n G_k(u_n)}{(u_n+\frac{1}{n})^\alpha}\,dx. \tag{3.4.6}$$

Since $\alpha > 1$ and $\dfrac{G_k(u_n)}{(u_n+\frac{1}{n})} \leqslant 1$, then

$$\int_\Omega \frac{h_n G_k(u_n)}{(u_n+\frac{1}{n})^\alpha}\,dx \leqslant \frac{1}{k^{\alpha-1}}\int_\Omega h\,dx.$$

Moreover, by the definition of G_k, it easily follows that

$$u_n G_k(u_n) = G_k^2(u_n) + kG_k(u_n),$$

and thus,

$$\begin{aligned}\frac{a_{N,s}}{2}\iint_Q \frac{|G_k(u_n(x)) - G_k(u_n(y))|^2}{|x-y|^{N+2s}}\,dx\,dy &- \lambda\int_\Omega \frac{G_k^2(u_n)}{|x|^{2s}}\,dx \\ &\leqslant \lambda k\int_\Omega \frac{G_k(u_n)}{|x|^{2s}}\,dx + \frac{1}{k^{\alpha-1}}\int_\Omega h\,dx.\end{aligned}$$

Since $\lambda < \Lambda_{N,s}$, by the Hardy inequality we conclude

$$\frac{a_{N,s}}{2}\left(1 - \frac{\lambda}{\Lambda_{N,s}}\right)\|G_k(u_n)\|^2_{H_0^s(\Omega)} \leqslant \lambda k\int_\Omega \frac{G_k^2(u_n)}{|x|^{2s}}\,dx + C(k,h). \tag{3.4.7}$$

On the other hand, by the Young inequality, for every $\varepsilon > 0$,

$$\lambda k \int_\Omega \frac{G_k(u_n)}{|x|^{2s}}\, dx \leqslant \frac{\lambda\varepsilon}{2} \int_\Omega \frac{G_k^2(u_n)}{|x|^{2s}}\, dx + \frac{k^2}{2\varepsilon} \int_\Omega \frac{dx}{|x|^{2s}}.$$

Plugging this into (3.4.7) and applying the Hardy inequality again we reach that

$$\|G_k(u_n)\|^2_{H_0^s} \leqslant C(k, \lambda, \Lambda_{N,s}, h),$$

and therefore $\{G_k(u_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $H_0^s(\Omega)$. Furthermore, this directly implies

$$\int_\Omega \frac{G_k^2(u_n)}{|x|^{2s}} dx \leqslant C(k, \lambda, \Lambda_{N,s}, h),$$

and thus

$$\int_\Omega \frac{u_n^2}{|x|^{2s}} dx = \int_\Omega \frac{T_k^2(u_n)}{|x|^{2s}}\, dx + \int_\Omega \frac{G_k^2(u_n)}{|x|^{2s}}\, dx \leqslant C(k, \lambda, \Lambda_{N,s}, h). \tag{3.4.8}$$

Likewise, testing in (3.4.2) with $T_k^\alpha(u_n)$, it follows

$$\begin{aligned}
&\frac{a_{N,s}}{2} \iint_Q \frac{(T_k^\alpha(u_n(x)) - T_k^\alpha(u_n(y)))(u_n(x) - u_n(y))}{|x-y|^{N+2s}} dx\, dy \\
&\leqslant \lambda \int_\Omega \frac{u_n T_k^\alpha(u_n)}{|x|^{2s}}\, dx + \int_\Omega \frac{h_n T_k^\alpha(u_n)}{(u_n + \frac{1}{n})^\alpha}\, dx \\
&\leqslant k^{\alpha-1}\lambda \int_\Omega \frac{u_n^2}{|x|^{2s}}\, dx + \int_\Omega h_n\, dx \\
&\leqslant C(k, \lambda, \Lambda_{N,s}, h),
\end{aligned}$$

where we have used (3.4.8) in the last line. Applying now Lemma 3.4.3 we conclude

$$\iint_Q \frac{(T_k^{\frac{\alpha+1}{2}}(u_n(x)) - T_k^{\frac{\alpha+1}{2}}(u_n(y)))^2}{|x-y|^{N+2s}}\, dx\, dy \leqslant C(\alpha, k, \lambda, \Lambda_{N,s}, h),$$

that is, $\{T_k^{\frac{\alpha+1}{2}}(u_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $H_0^s(\Omega)$.

Claim.- $\{T_k(u_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $H^s_{loc}(\Omega)$.

Since $\{u_n\}_{n\in\mathbb{N}}$ is an increasing sequence, necessarily $T_k(u_n) \geqslant T_k(u_1)$. Moreover, by Lemma 3.4.1, for all $\Omega' \subset\subset \Omega$, there exists a positive constant $c_{\Omega'}$ such that $u_1 \geqslant C(\Omega')$, and thus

$$T_k(u_n) \geqslant \min\{k, c_{\Omega'}\} =: C_0.$$

Let $(x, y) \in \Omega' \times \Omega'$, and define

$$\eta_n := \frac{T_k(u_n(x))}{C_0} \text{ and } \beta_n := \frac{T_k(u_n(y))}{C_0}.$$

Clearly $\eta_n, \beta_n \geqslant 1$, and hence the following inequality holds,

$$(\eta_n - \beta_n)^2 \leqslant (\eta_n^{\frac{\alpha+1}{2}} - \beta_n^{\frac{\alpha+1}{2}})^2. \tag{3.4.9}$$

Indeed, if $\eta_n = \beta_n$, the estimate is trivial.
Without loss of generality we may assume $\eta_n > \beta_n \geqslant 1$, and hence $0 < x := \frac{\beta_n}{\eta_n} < 1$. Since $\alpha > 1$, we easily obtain

$$0 \leqslant 1 - x \leqslant 1 - x^{\frac{\alpha+1}{2}},$$

and hence

$$(1-x)^2 \leqslant (1 - x^{\frac{\alpha+1}{2}})^2.$$

Furthermore, $\eta_n^2 < \eta_n^{\alpha+1}$, and thus

$$\eta_n^2(1-x)^2 \leqslant \eta_n^{\alpha+1}(1 - x^{\frac{\alpha+1}{2}})^2,$$

and (3.4.9) holds. Recalling the definition of η_n and β_n, we conclude that for (x, y) in $\Omega' \times \Omega'$, we have

$$(T_k(u_n(x)) - T_k(u_n(y)))^2 \leqslant C_0^{1-\alpha}(T_k^{\frac{\alpha+1}{2}}(u_n(x)) - T_k^{\frac{\alpha+1}{2}}(u_n(y)))^2.$$

Thus, since $\{T_k^{\frac{\alpha+1}{2}}(u_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $H_0^s(\Omega)$,

$$\iint_{\Omega'\times\Omega'} \frac{(T_k(u_n(x)) - T_k(u_n(y)))^2}{|x-y|^{N+2s}} dx\, dy \leqslant C(\Omega', k, \lambda, \Lambda_{N,s}, h),$$

and the claim is proved (notice that the L^2 term in the norm of $H^s_{loc}(\Omega)$ is bounded as a straigthforward consequence of (3.4.8)).

Combining the three uniform estimates, we conclude the existence of $u \in H^s_{loc}(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } H^s_{loc}(\Omega), \\ G_k(u_n) &\rightharpoonup G_k(u) && \text{in } H^s_0(\Omega), \text{ and} \\ T_k^{\frac{\alpha+1}{2}}(u) &\rightharpoonup T_k^{\frac{\alpha+1}{2}}(u) && \text{in } H^s_0(\Omega). \end{aligned}$$

Since $\alpha > 1$, in particular this implies

$$u_n \to u \text{ in } L^r(\Omega), \text{ for every } 1 \leqslant r < 2_s^*.$$

Let $\phi \in \widetilde{\mathcal{T}}$, where $\widetilde{\mathcal{T}}$ was defined in (3.1.2). Testing in (3.4.2), it yields

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n (-\Delta)^{s/2}\phi\, dx = \lambda \int_\Omega \frac{u_n \phi}{|x|^{2s}}\, dx + \int_\Omega \frac{\phi h_n(x)}{\left(u_n + \frac{1}{n}\right)^\alpha}\, dx. \tag{3.4.10}$$

By Lemma 3.4.1 and convergences of u_n,

$$\lambda \int_\Omega \frac{u_n\phi}{|x|^{2s}} + \int_\Omega \frac{\phi h_n}{\left(u_n + \frac{1}{n}\right)^\alpha} dx \to \lambda \int_\Omega \frac{u\phi}{|x|^{2s}}\, dx + \int_\Omega \frac{\phi h}{u^a} dx < +\infty.$$

Moreover,

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n (-\Delta)^{s/2} \phi \, dx = \int_\Omega u_n (-\Delta)^s \phi \, dx \to \int_\Omega u (-\Delta)^s \phi \, dx,$$

and thus u is a weak solution of (3.4.1).

Finally, using u_n^α as a test function in (3.4.2), it follows

$$\frac{a_{N,s}}{2} \iint_Q \frac{(u_n(x) - u_n(y))(u_n^\alpha(x) - u_n^\alpha(y))}{|x-y|^{N+2s}} \, dx \, dy \leqslant \lambda \int_\Omega \frac{u_n^{\alpha+1}}{|x|^{2s}} \, dx + \int_\Omega h_n \, dx.$$

By Lemma 3.4.3 and the Hardy inequality,

$$\frac{a_{N,s}}{2} \left(\frac{4\alpha}{(\alpha+1)^2} - \frac{\lambda}{\Lambda_{N,s}} \right) \iint_Q \frac{(u_n^{\frac{\alpha+1}{2}}(x) - u_n^{\frac{\alpha+1}{2}}(y))^2}{|x-y|^{N+2s}} \, dx \, dy \leqslant \|h\|_{L^1(\Omega)}.$$

Therefore, if

$$\left(\frac{4\alpha}{(\alpha+1)^2} - \frac{\lambda}{\Lambda_{N,s}} \right) > 0,$$

we conclude that

$$\|u^{\frac{\alpha+1}{2}}\|_{H_0^s(\Omega)} \leqslant \liminf_{n \to \infty} \|u_n^{\frac{\alpha+1}{2}}\|_{H_0^s(\Omega)} \leqslant C,$$

with C a positive constant independent of n. □

Finally, to conclude this chapter, to complement the results obtained in Proposition 3.4.2, it is worth to wonder what happens if, instead of asking $h \in L^{(2_s^*)'}(\Omega)$, we have less regularity. In particular, supported by the analysis that we performed in Chapter 2, we will try to study the problem from the point of view of the weighted operator

$$L_\gamma(v(x)) := a_{N,s} \text{ p.v.} \int_{\mathbb{R}^N} (v(x) - v(y)) K(x,y) dy,$$

already defined in (2.2.4), with (see Lemma 1.2.3 and Remark 1.2.4 for more information about γ)

$$K(x,y) := \frac{1}{|x|^\gamma} \frac{1}{|y|^\gamma} \frac{1}{|x-y|^{N+2s}}, \quad 0 < \gamma := \frac{N-2s}{2} - \alpha < \frac{N-2s}{2}.$$

Let us recall that this operator allowed us to join the Laplacian and the Hardy potential. More precisely, if we have that $u \in H_0^s(\Omega)$ satisfies

$$\begin{cases} (-\Delta)^s u - \lambda \dfrac{u}{|x|^{2s}} = f(x,u) & \text{in } \Omega, \\ \qquad\qquad u > 0 & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $v(x) := |x|^\gamma u(x)$ solves

$$\begin{cases} L_\gamma(v) = |x|^{-\gamma} f(x,u) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Moreover, let us recall that we denoted

$$d\mu := \frac{dx}{|x|^{2\gamma}}, \quad \text{and} \quad d\nu := K(x,y)dxdy.$$

Thus, attending to the form of the new problem, the question for us is whether we are able to find a solution to problem (3.4.1) when we ask $h \in L^1(\Omega, d\mu)$. First of all, we define the Hilbert space $\tilde{Y}^{s,\gamma}(\Omega)$ as

$$\tilde{Y}^{s,\gamma}(\Omega) := \left\{ \phi \in L^2(\Omega, d\mu) : \iint_Q |\phi(x) - \phi(y)|^2 \, d\nu < +\infty \right\},$$

endowed with the norm

$$\|\phi\|_{\tilde{Y}^{s,\gamma}(\Omega)} := \left(\int_\Omega |\phi(x)|^2 \, d\mu + \iint_Q |\phi(x) - \phi(y)|^2 \, d\nu \right)^{\frac{1}{2}},$$

and we define the space $\tilde{Y}_0^{s,\gamma}(\Omega)$ as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to this norm. In particular, we denote

$$\|\phi\|_{\tilde{Y}_0^{s,\gamma}(\Omega)} := (\langle L_\gamma \phi, \phi \rangle)^{\frac{1}{2}} = \left(\iint_Q |\phi(x) - \phi(y)|^2 \, d\nu \right)^{\frac{1}{2}}.$$

Remark 3.4.5. *Notice that the spaces $\tilde{Y}^{s,\gamma}(\Omega)$ and $\tilde{Y}_0^{s,\gamma}(\Omega)$ are different from $Y^{s,\gamma}(\Omega)$ and $Y_0^{s,\gamma}(\Omega)$ defined in (2.2.8). In particular, the double integrals on those spaces were defined over $\Omega \times \Omega$, that is smaller than the domain Q, where we integrate here.*

We will also need the following compactness result.

Lemma 3.4.6. *Let $\{u_n\}_{n\in\mathbb{N}} \subset \tilde{Y}_0^{s,\gamma}(\Omega)$ be an increasing sequence of nonnegative functions such that $L_\gamma(u_n) \geqslant 0$, and assume that $u_n \rightharpoonup u$ weakly in $\tilde{Y}_0^{s,\gamma}(\Omega)$. Then $u_n \to u$ strongly in $\tilde{Y}_0^{s,\gamma}(\Omega)$.*

Proof. By the weak convergence, we know that also $u_n \to u$ a.e. in Ω. Moreover, since u_n is increasing, necessarily

$$u_n - u \leqslant 0 \text{ a.e. in } \Omega.$$

Therefore, using that $L_\gamma(u_n) \geqslant 0$, we find that

$$\langle L_\gamma(u_n), u_n - u \rangle \leqslant 0.$$

On the other hand, by the weak convergence

$$\langle L_\gamma(u), u_n - u\rangle \to 0 \text{ as } n \to \infty,$$

and then, thanks to the linearity of $L_{\gamma,\Omega}$,

$$\begin{aligned} \|u_n - u\|^2_{\tilde{Y}_0^{s,\gamma}(\Omega)} &= \langle L_\gamma(u_n), u_n - u\rangle - \langle L_\gamma(u), u_n - u\rangle \\ &\leqslant \langle L_\gamma(u), (u_n - u)\rangle \to 0 \text{ as } n \to +\infty. \end{aligned}$$

□

Thus, we can finally state the existence result.

Theorem 3.4.7. *Assume $\alpha < 1$, $\lambda < \Lambda_{N,s}$, and $h \in L^1(\Omega, |x|^{-(1-\alpha)\gamma})$. Then the problem* (3.4.1) *has at least a weak solution.*

Proof. Let u_n be the unique positive solution to (3.4.2). By setting $v_n := |x|^\gamma u_n$, it follows that v_n satisfies

$$\text{(3.4.11)} \qquad \begin{cases} L_\gamma(v_n) \leqslant |x|^{-(1-\alpha)\gamma}\dfrac{h_n}{v_n^\alpha} & \text{in } \Omega, \\ v_n > 0 & \text{in } \Omega, \\ v_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Testing here with v_n^α we obtain

$$\frac{4a}{(a+1)^2}\iint_Q \frac{(v_n^{\frac{\alpha+1}{2}}(x) - v_n^{\frac{a+1}{2}}(y))^2}{|x-y|^{N+2s}}\,d\nu \leqslant \int_\Omega \frac{h_n}{|x|^{(1-\alpha)\gamma}}\,dx \leqslant C,$$

with $C > 0$ independent of n. Thus we conclude that the sequence $\{v_n^{\frac{\alpha+1}{2}}\}_{n\in\mathbb{N}}$ is uniformly bounded in $\tilde{Y}_0^{s,\gamma}(\Omega)$. Hence, there exists $v \in \tilde{Y}_0^{s,\gamma}(\Omega)$ such that

$$v_n^{\frac{\alpha+1}{2}} \rightharpoonup v^{\frac{\alpha+1}{2}} \text{ in } \tilde{Y}_0^{s,\gamma}(\Omega).$$

On the other hand, since $\frac{\alpha+1}{2} < 1$, $L_\gamma(v_n) \geqslant 0$ implies that $L_\gamma(v_n^{\frac{\alpha+1}{2}}) \geqslant 0$. This, together with the weak convergence, the monotonicity of v_n and Lemma 3.4.6, allows us to conclude

$$v_n^{\frac{\alpha+1}{2}} \to v^{\frac{\alpha+1}{2}} \text{ in } \tilde{Y}_0^{s,\gamma}(\Omega).$$

Passing to the limit in (3.4.11), we deduce that v solves

$$\begin{cases} L_\gamma(v) = |x|^{-(1-\alpha)\gamma}\dfrac{h}{v^\alpha} & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

in the weak sense. Defining now $u := |x|^{-\gamma}v$, then $\lambda\dfrac{u}{|x|^{2s}} \in L^1(\Omega)$ and u is a weak solution of (3.4.1). □

PART II

Bifurcation results for a critical nonlocal equation in $\mathbb{R}^N$

Chapter 4

Multiplicity of solutions via a Lyapunov-Schdmidt reduction

In this chapter we deal with the problem

$$(4.0.1)\qquad (-\Delta)^s u = \varepsilon\, h\, u^q + u^p \ \text{ in } \mathbb{R}^N, \qquad u > 0,$$

where $N > 4s$, $\varepsilon > 0$ is a small parameter, $p = \frac{N+2s}{N-2s}$ is the fractional critical Sobolev exponent, $0 < q < p$ and h is a continuous function that satisfies

$$(4.0.2)\qquad \omega := supp\, h \text{ is compact}$$

$$(4.0.3)\qquad \text{and} \qquad h_+ \not\equiv 0.$$

There exists a huge literature concerning the search of solutions for this kind of perturbative problems in the classical case, i.e. when $s = 1$ and the fractional Laplacian boils down to the classical Laplacian, see [17, 18, 19, 20, 21, 37, 61, 69, 76, 137, 138]. In particular, Theorem 4.1.2 here can be seen as the nonlocal counterpart of [18, Theorem 1.3]. See also [110], where the concave term appears for the first time.

In the fractional case, the situation is more involved. Namely, the nonlocal Schrödinger equation has recently received a growing attention not only for the challenging mathematical difficulties that it offers, but also due to some important physical applications (see e.g. [131], the appendix in [78], and the references therein). In the subcritical case, this nonlocal Schrödinger equation can be written as

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = u^p \ \text{ in } \mathbb{R}^N,$$

with $1 < p < \frac{N+2s}{N-2s}$ and V a smooth potential. Multi-peak solutions for this type of equations were considered recently in [80]. In this case, a key ingredient in the proof is the uniqueness and nondegeneracy of the ground state solution of the corresponding unperturbed problem, which has been proved in [105] for any $s \in (0, 1)$ and in any dimension, after previous works in dimension 1 (see [104]) and for s close to 1 (see [94]).
Moreover, given a bounded domain $\Omega \subset \mathbb{R}^N$, the Dirichlet problem

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + u = u^p & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

was considered in [78], where the authors construct solutions that concentrate at the interior of the domain.

Concentrating solutions for fractional problems involving critical or almost critical exponents were considered in [68]. See also [64] for some concentration phenomena in particular cases and [153] for the study of the soliton dynamics in related problems. See also [65] for a semilinear problem with critical power, related to the scalar curvature problem, that also exploits a Lyapunov-Schmidt reduction. It is worth pointing out that, in our case, the presence of the subcritical, possibly sublinear, power in our problem introduces extra difficulties that have required the development of certain elliptic regularity theory, and the careful analysis of the corresponding functional framework.

The results of this chapter can be found in [84].

4.1 Preliminaries and functional setting.

The strategy to find solutions of problem (4.0.1) will be to consider it as a perturbation of the equation

$$(-\Delta)^s u = u^p \quad \text{in } \mathbb{R}^N, \tag{4.1.1}$$

with $p = 2_s^* - 1$. It is known that the minimizers of the Sobolev embedding in $\mathbb{R}^N$ are unique, up to translations and positive dilations. Namely if we set

$$z_0(x) := \alpha_{N,s} \frac{1}{(1+|x|^2)^{(N-2s)/2}}, \tag{4.1.2}$$

then all the minimizers of the Sobolev embedding are obtained by the formula

$$z_{\mu,\xi}(x) := \mu^{(2s-N)/2}\, z_0\left(\frac{x-\xi}{\mu}\right), \tag{4.1.3}$$

where $\mu > 0$, $\xi \in \mathbb{R}^N$. The normalizing constant $\alpha_{N,s}$ depends only on N and s (see [79, 136, 169] and the rerenfences therein). Notice also that equation (4.1.1) is the Euler-Lagrange equation of this Sobolev embedding minimization problem.

Furthermore, it has been showed in [79] that solutions to (4.1.1) of the form (4.1.3) are nondegenerate. That is, setting $\partial_\mu z_{\mu,\xi}$ and $\partial_\xi z_{\mu,\xi}$ the derivative of $z_{\mu,\xi}$ with respect to the parameters μ and ξ respectively, then all bounded solutions of the linear equation

$$(-\Delta)^s \psi = p\, z_{\mu,\xi}^{p-1}\, \psi \quad \text{in } \mathbb{R}^N,$$

are linear combinations of $\partial_\mu z_{\mu,\xi}$ and $\partial_\xi z_{\mu,\xi}$. We also refer to [103], where the nondegeneracy result was proved in detail for $s = 1/2$ and $N = 3$ (but the proof can be extended in higher dimensions and for fractional exponents $s \in (0, N/2)$ as well).

We set

$$[u]^2_{\dot{H}^s(\mathbb{R}^N)} := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy,$$

and we define the space $\dot{H}^s(\mathbb{R}^N)$ as the completion of the space of smooth and rapidly deceasing functions (the so-called Schwartz space) with respect to the norm

$$[u]_{\dot{H}^s(\mathbb{R}^N)} + \|u\|_{L^{2_s^*}(\mathbb{R}^N)},$$

where 2_s^* is the fractional critical exponent.

We also introduce the space

$$X^s := \dot{H}^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

equipped with the norm

$$\|u\|_{X^s} := [u]_{\dot{H}^s(\mathbb{R}^N)} + \|u\|_{L^\infty(\mathbb{R}^N)}.$$

Indeed, along this chapter we will work with the following concept of solution.

Definition 4.1.1. Given $f \in L^\beta(\mathbb{R}^N)$, where $\beta := \frac{2N}{N+2s}$, we say that $u \in X^s$ is an energy solution to $(-\Delta)^s u = f$ in $\mathbb{R}^N$ if

$$\iint_{\mathbb{R}^{2n}} \frac{\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s}}\,dx\,dy = \int_{\mathbb{R}^N} f\,\varphi\,dx,$$

for any $\varphi \in X^s$.

In particular, the aim will be to prove the next existence result.

Theorem 4.1.2. *Suppose that h is a continuous function that satisfies* (4.0.2) *and* (4.0.3). *Then, there exist $\varepsilon_0 > 0$, $\mu_1 > 0$ and $\xi_1 \in \mathbb{R}^N$ such that problem* (4.0.1) *has a positive solution $u_{1,\varepsilon}$ for any $\varepsilon \in (0,\varepsilon_0)$, and $u_{1,\varepsilon} \to z_{\mu_1,\xi_1}$ in X^s as $\varepsilon \to 0$.*

Also, if h changes sign, then for any $\varepsilon \in (0,\varepsilon_0)$ there exists a second positive solution $u_{2,\varepsilon}$ to (4.0.1) *that, as $\varepsilon \to 0$, converges in X^s to z_{μ_2,ξ_2} with $\mu_2 > 0$, $\mu_2 \neq \mu_1$, and $\xi_2 \in \mathbb{R}^N$, $\xi_2 \neq \xi_1$.*

In order to prove Theorem 4.1.2 we will use a Lyapunov-Schmidt reduction, that takes advantage of the variational structure of the problem. Indeed, positive solutions to (4.0.1) can be found as critical points of the functional $\mathcal{I}_\varepsilon : X^s \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{I}_\varepsilon(u) :=& \frac{a_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &- \frac{\varepsilon}{q+1}\int_{\mathbb{R}^N} h(x)\,u_+^{q+1}(x)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} u_+^{p+1}(x)\,dx.\end{aligned} \tag{4.1.4}$$

We notice that $\mathcal{I}_\varepsilon$ can be written as

$$\mathcal{I}_\varepsilon(u) = \mathcal{I}_0(u) - \varepsilon\,\mathcal{I}(u), \tag{4.1.5}$$

where

$$\mathcal{I}_0(u) := \frac{a_{N,s}}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{p+1}\int_{\mathbb{R}^N} u_+^{p+1}(x)\,dx, \tag{4.1.6}$$

called the unperturbed functional, and

$$\mathcal{I}(u) := \frac{1}{q+1} \int_{\mathbb{R}^N} h(x)\, u_+^{q+1}(x)\, dx. \tag{4.1.7}$$

Indeed, we will use a perturbation method that allows us to find critical points of $\mathcal{I}_\varepsilon$ by bifurcating from a manifold of critical points of $\mathcal{I}_0$ (see for instance [22] for the abstract method).

Notice that critical points of $\mathcal{I}_0$ are solutions to (4.1.1), and so, in order to construct solutions to (4.0.1), we will start from functions of the form (4.1.3) and we will add a small error to them in such a way that we obtain solutions to the perturbed problem.

This small error will be found by means of the Implicit Function Theorem. To do this, a crucial ingredient will be the nondegeneracy condition proved in [79] for $z_{\mu,\xi}$, but the application of the linear theory in our case is non-standard and it requires a pointwise control of the functional spaces.

Roughly speaking, one additional difficulty for us is indeed that when $q < 1$ the energy functional is not smooth at the zero level set, and so the classical Implicit Function Theorem cannot be applied, unless we can avoid the singularity. For this, the classical Hilbert space framework is not enough, and we have to keep track of the pointwise behavior of the functions inside our functional analysis framework. This is for instance the main reason for which we work in the more robust space X^s rather than in the more classical space $\dot{H}^s(\mathbb{R}^N)$.

Of course, the change of functional setting produces some difficulties in the invertibility of the operators, since the Hilbert-Fredholm theory does not directly apply, and we will have to compensate it by an appropriate elliptic regularity theory.

Once these difficulties are overcome, the Lyapunov-Schmidt reduction allows us to reduce our problem to the one of finding critical points of the perturbation $\mathcal{I}$, introduced in (4.1.7). For this, we set

$$\Gamma(\mu,\xi) := \mathcal{I}(z_{\mu,\xi}), \tag{4.1.8}$$

where $z_{\mu,\xi}$ has been introduced in (4.1.3). The study of the behavior of Γ will give us the existence of critical points of $\mathcal{I}$, and so the existence of solution to (4.0.1).

In particular, we perform here a detailed analysis of the linearized equation, that is the key ingredient to use the Lyapunov-Schmidt arguments.

4.1.1 Fractional elliptic estimates.

Here we obtain some uniform elliptic estimates on Riesz potential that will be used in Section 4.2 in order to obtain the continuity properties of our functionals.

We recall that

$$H^s(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable s.t. } \|u\|_{L^2(\mathbb{R}^N)} + [u]_{\dot{H}^s(\mathbb{R}^N)} < +\infty\}.$$

To start, we point out that the fractional Sobolev inequality holds in X^s, thanks to a simple limit procedure.

Lemma 4.1.3. *Let $N > 2s$. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a measurable function. Suppose that there exists a sequence of functions $f_k \in H^s(\mathbb{R}^N)$ such that $f_k \to f$ in $\dot{H}^s(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Then*

$$\|f\|_{L^{2^*_s}(\mathbb{R}^N)} \leqslant C\,[f]_{\dot{H}^s(\mathbb{R}^N)}, \tag{4.1.9}$$

for some $C > 0$ depending on N and s. In particular, the inequality in (4.1.9) holds true for any $f \in X^s$.

Proof. For each $k \in \mathbb{N}$, we have that $f_k \in H^s(\mathbb{R}^N)$, so we can apply the fractional Sobolev inequality (see Theorem 0.0.4) to obtain

$$\|f_k\|_{L^{2^*_s}(\mathbb{R}^N)} \leqslant C\,[f_k]_{\dot{H}^s(\mathbb{R}^N)}. \tag{4.1.10}$$

Since

$$\lim_{k\to+\infty}[f_k]_{\dot{H}^s(\mathbb{R}^N)} \leqslant \lim_{k\to+\infty}[f_k - f]_{\dot{H}^s(\mathbb{R}^N)} + [f]_{\dot{H}^s(\mathbb{R}^N)} \leqslant [f]_{\dot{H}^s(\mathbb{R}^N)}$$

and, by Fatou's Lemma,

$$\begin{aligned}\liminf_{k\to+\infty}\|f_k\|_{L^{2^*_s}(\mathbb{R}^N)} &= \left[\liminf_{k\to+\infty}\int_{\mathbb{R}^N}|f_k(x)|^{2^*_s}\,dx\right]^{1/2^*_s}\\ &\geqslant \left[\int_{\mathbb{R}^N}|f(x)|^{2^*_s}\,dx\right]^{1/2^*_s} = \|f\|_{L^{2^*_s}(\mathbb{R}^N)},\end{aligned}$$

we can pass to the limit in (4.1.10) and obtain (4.1.9). □

Here we state the fractional elliptic regularity needed for our goals.

Theorem 4.1.4. *Let $N > 4s$. Let $\beta := 2N/(N+2s)$ and $\psi \in L^\beta(\mathbb{R}^N)$. Let also*

$$J\psi(x) := \int_{\mathbb{R}^N}\frac{\psi(y)}{|x-y|^{N-2s}}\,dy. \tag{4.1.11}$$

Then,

(4.1.12) $J\psi \in L^{2^*_s}(\mathbb{R}^N)$, *and* $\|J\psi\|_{L^{2^*_s}(\mathbb{R}^N)} \leqslant C\,\|\psi\|_{L^\beta(\mathbb{R}^N)}$;

(4.1.13) $J\psi \in \dot{H}^s(\mathbb{R}^N)$, *and* $[J\psi]_{\dot{H}^s(\mathbb{R}^N)} \leqslant C\,\|\psi\|_{L^\beta(\mathbb{R}^N)}$;

(4.1.14) $(-\Delta)^s(J\psi) = c\psi$ *in the energy sense, i.e.*

$$\iint_{\mathbb{R}^{2N}}\frac{\big((J\psi)(x)-(J\psi)(y)\big)\big(\phi(x)-\phi(y)\big)}{|x-y|^{N+2s}}\,dx\,dy = c\int_{\mathbb{R}^N}\psi(x)\,\phi(x)\,dx$$

for any $\phi \in X^s$;

(4.1.15) *if, in addition, it holds that $\psi \in L^\infty(\mathbb{R}^N)$, then $J\psi \in L^\infty(\mathbb{R}^N)$,*

$$\text{and } \|J\psi\|_{L^\infty(\mathbb{R}^N)} \leqslant C\left(\|\psi\|_{L^\infty(\mathbb{R}^N)} + \|\psi\|_{L^\beta(\mathbb{R}^N)}\right).$$

Here above, C and c are suitable positive constants only depending on N and s.

Remark 4.1.5. *In the sequel, for simplicity we will just take $c = 1$ in (4.1.14). This can be accomplished simply by renaming J to $c^{-1}J$.*

Proof. The claim in (4.1.12) follows from an appropriate version of the Hardy-Littlewood-Sobolev inequality, namely [162, Theorem 1, page 119], used here with $\alpha := 2s$, $p := \beta$ and $q := 2^*_s$.

We take now a sequence of smooth and rapidly decreasing functions ψ_j that converge to ψ in $L^\beta(\mathbb{R}^N)$, and we set $\Psi_j := J\psi_j$. We also set $\Psi := J\psi$. Thus, by (4.1.12), we have that

$$\|\Psi_j - \Psi\|_{L^{2^*_s}(\mathbb{R}^N)} = \|J(\psi_j - \psi)\|_{L^{2^*_s}(\mathbb{R}^N)} \leqslant C\|\psi_j - \psi\|_{L^\beta(\mathbb{R}^N)} \to 0$$

as $j \to +\infty$. Thus, up to a subsequence,

$$\Psi_j \to \Psi \text{ a.e. in } \mathbb{R}^N. \tag{4.1.16}$$

Moreover, by [162, Lemma 2(b)], we have that

$$\int_{\mathbb{R}^N} (J\psi_j)(x)\,\overline{g(x)}\,dx = c\int_{\mathbb{R}^N} \hat\psi_j(\xi)\,|\xi|^{-2s}\,\overline{\hat g(\xi)}\,d\xi, \tag{4.1.17}$$

for some $c > 0$, for every g that is smooth and rapidly decreasing (and possibly complex valued). As standard, we have denoted by $\hat g := \mathcal{F}(g)$ the Fourier transform of g.

Now, for any ϕ smooth and rapidly decreasing and any $\delta > 0$, we take g_δ to be the inverse Fourier transform of $(|\xi|^2 + \delta)^s\hat\phi$, in symbols $g_\delta := \mathcal{F}^{-1}\big((|\xi|^2 + \delta)^s\hat\phi\big)$. We remark that $(|\xi|^2 + \delta)^s\hat\phi$ is smooth and rapidly decreasing, hence so is g_δ. Accordingly, (4.1.17) implies that

$$\int_{\mathbb{R}^N} (J\psi_j)(x)\,\overline{g_\delta(x)}\,dx = c\int_{\mathbb{R}^N} \hat\psi_j(\xi)\,|\xi|^{-2s}\,\overline{(|\xi|^2 + \delta)^s\hat\phi(\xi)}\,d\xi. \tag{4.1.18}$$

We claim that

$$g_\delta \to \mathcal{F}^{-1}(|\xi|^{2s}\hat\phi) \text{ in } L^2(\mathbb{R}^N), \text{ as } \delta \to 0. \tag{4.1.19}$$

To check this, we use Plancherel Theorem to compute

$$\begin{aligned}\|g_\delta - \mathcal{F}^{-1}(|\xi|^{2s}\hat\phi)\|^2_{L^2(\mathbb{R}^N)} &= \|\hat g_\delta - |\xi|^{2s}\hat\phi\|^2_{L^2(\mathbb{R}^N)}\\ &= \big\|[(|\xi|^2 + \delta)^s - |\xi|^{2s}]\,\hat\phi\big\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \big|(|\xi|^2 + \delta)^s - |\xi|^{2s}\big|^2\,|\hat\phi(\xi)|^2\,d\xi.\end{aligned} \tag{4.1.20}$$

Then we observe that, if $\delta \in (0, 1)$,

$$\big|(|\xi|^2 + \delta)^s - |\xi|^{2s}\big|^2 \leqslant 4(|\xi|^2 + 1)^{2s},$$

and the function $\xi \mapsto (|\xi|^2 + 1)^{2s}\,|\hat\phi(\xi)|^2$ belongs to $L^1(\mathbb{R}^N)$, since $\hat\phi$ is also rapidly decreasing, thus (4.1.19) follows from (4.1.20) and the Dominated Convergence Theorem.

Moreover, since ψ_j is rapidly decreasing, a direct computation with convolutions (see e.g. [78, Lemma 5.1]) gives that

$$|J\psi_j(x)| \leqslant \frac{C_j}{1+|x|^{N-2s}}, \tag{4.1.21}$$

for some $C_j > 0$. In particular, since $N > 4s$, we have that

$$\Psi_j = J\psi_j \in L^2(\mathbb{R}^N). \tag{4.1.22}$$

As a matter of fact, the derivatives of ψ_j are rapidly decreasing as well and $\nabla\Psi_j = J(\nabla\psi_j)$, thus the argument above also shows that $\nabla\Psi_j \in L^2(\mathbb{R}^N, \mathbb{R}^N)$, and so

$$\Psi_j \in H^1(\mathbb{R}^N). \tag{4.1.23}$$

Using (4.1.19), (4.1.22) and the Plancherel Theorem, we conclude that

$$\begin{aligned} \lim_{\delta\to 0}\int_{\mathbb{R}^N} (J\psi_j)(x)\,\overline{g_\delta(x)}\,dx &= \int_{\mathbb{R}^N} \Psi_j(x)\,\overline{\mathcal{F}^{-1}(|\xi|^{2s}\hat\phi)(x)}\,dx \\ &= \int_{\mathbb{R}^N} \hat\Psi_j(\xi)\,\overline{|\xi|^{2s}\hat\phi(\xi)}\,d\xi = \int_{\mathbb{R}^N} |\xi|^{2s}\hat\Psi_j(\xi)\,\overline{\hat\phi(\xi)}\,d\xi. \end{aligned} \tag{4.1.24}$$

Now we point out that, for $\delta \in (0,1)$,

$$\left| |\xi|^{-2s}\,\overline{(|\xi|^2+\delta)^s\hat\phi(\xi)} \right| \leqslant |\xi|^{-2s}\,(|\xi|^2+1)^s|\hat\phi(\xi)|,$$

and this function is in $L^1(\mathbb{R}^N)$, since $N > 2s$. Accordingly, the Dominated Convergence Theorem gives that

$$\lim_{\delta\to 0}\int_{\mathbb{R}^N} \hat\psi_j(\xi)\,|\xi|^{-2s}\,\overline{(|\xi|^2+\delta)^s\hat\phi(\xi)}\,d\xi = \int_{\mathbb{R}^N} \hat\psi_j(\xi)\,\overline{\hat\phi(\xi)}\,d\xi.$$

This, (4.1.18) and (4.1.24) imply that

$$\int_{\mathbb{R}^N} |\xi|^{2s}\hat\Psi_j(\xi)\,\overline{\hat\phi(\xi)}\,d\xi = c\int_{\mathbb{R}^N} \hat\psi_j(\xi)\,\overline{\hat\phi(\xi)}\,d\xi = c\int_{\mathbb{R}^N} \psi_j(x)\,\phi(x)\,dx, \tag{4.1.25}$$

for any ϕ smooth and rapidly decreasing.

Now we fix $j \in \mathbb{N}$ and make use of (4.1.23): accordingly, by density, we find a sequence $\Psi_{j,k}$ of smooth and rapidly decreasing functions that converge to Ψ_j in $H^1(\mathbb{R}^N)$ as $k \to +\infty$.

In particular, $\Psi_{j,k} \to \Psi_j$ in $L^2(\mathbb{R}^N)$ and so, by Plancherel Theorem, also $\hat\Psi_{j,k} \to \hat\Psi_j$ in $L^2(\mathbb{R}^N)$, as $k \to +\infty$. Moreover, $|\xi|^{2s} \leqslant 1$ if $|\xi| \leqslant 1$ and $|\xi|^{2s} \leqslant |\xi|^2$ if $|\xi| \geqslant 1$, thus

$$|\xi|^{2s} \leqslant 1 + |\xi|^2. \tag{4.1.26}$$

Consequently,

$$\begin{aligned} \int_{\mathbb{R}^N} |\xi|^{2s}\big|\hat\Psi_{j,k}(\xi) - \hat\Psi_j(\xi)\big|^2\,d\xi &\leqslant \int_{\mathbb{R}^N} (1+|\xi|^2)\,\big|\mathcal{F}\big(\Psi_{j,k}(\xi) - \Psi_j(\xi)\big)\big|^2\,d\xi \\ &\leqslant C\|\Psi_{j,k} - \Psi_j\|^2_{H^1(\mathbb{R}^N)} \to 0 \end{aligned} \tag{4.1.27}$$

as $k \to +\infty$, and therefore

$$\lim_{k\to+\infty} \int_{\mathbb{R}^N} |\xi|^{2s}\hat{\Psi}_j(\xi)\,\overline{\hat{\Psi}_{j,k}(\xi)}\,d\xi = \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{\Psi}_j(\xi)|^2\,d\xi.$$

Then we apply (4.1.25) with $\phi := \Psi_{j,k}$, and therefore we see that

$$\begin{aligned}\int_{\mathbb{R}^N} |\xi|^{2s}|\hat{\Psi}_j(\xi)|^2\,d\xi &= \lim_{k\to+\infty} \int_{\mathbb{R}^N} |\xi|^{2s}\hat{\Psi}_j(\xi)\,\overline{\hat{\Psi}_{j,k}(\xi)}\,d\xi \\ &= \lim_{k\to+\infty} c\int_{\mathbb{R}^N} \hat{\psi}_j(\xi)\,\overline{\hat{\Psi}_{j,k}(\xi)}\,d\xi = c\int_{\mathbb{R}^N} \hat{\psi}_j(\xi)\,\overline{\hat{\Psi}_j(\xi)}\,d\xi.\end{aligned}$$

Thus, by the Hölder inequality with exponents β and $2N/(N-2s)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\xi|^{2s}|\hat{\Psi}_j(\xi)|^2\,d\xi &= c\int_{\mathbb{R}^N} \psi_j(\xi)\,\Psi_j(\xi)\,d\xi \\ &\leqslant c\,\|\psi_j\|_{L^\beta(\mathbb{R}^N)}\,\|\Psi_j\|_{L^{2^*_s}(\mathbb{R}^N)} \leqslant C\,\|\psi_j\|^2_{L^\beta(\mathbb{R}^N)},\end{aligned}$$

where (4.1.12) was used in the last step.

This (together with the equivalence of the seminorm in $H^s(\mathbb{R}^N)$, see [82, Proposition 3.4]) says that

$$\iint_{\mathbb{R}^{2N}} \frac{|\Psi_j(x)-\Psi_j(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \leqslant C\|\psi_j\|^2_{L^\beta(\mathbb{R}^N)}.$$

So we recall (4.1.16) and we take the limit as $j \to +\infty$, obtaining, by Fatou's Lemma and the fact that $\psi_j \to \psi$ in $L^\beta(\mathbb{R}^N)$, that

$$\iint_{\mathbb{R}^{2N}} \frac{|\Psi(x)-\Psi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \leqslant C\|\psi\|^2_{L^\beta(\mathbb{R}^N)},$$

that establishes the estimate in (4.1.13).

Now we show that $\Psi = J\psi \in \dot{H}^s(\mathbb{R}^N)$. For this, we notice that, since $\psi \in L^\beta(\mathbb{R}^N)$, there exists a sequence of smooth and rapidly decreasing functions ψ_j such that ψ_j converges to ψ in $L^\beta(\mathbb{R}^N)$ as $j \to +\infty$. So, thanks to the estimates in (4.1.12) and (4.1.13), we have that

$$\|J\psi - J\psi_j\|_{L^{2^*_s}(\mathbb{R}^N)} = \|J(\psi-\psi_j)\|_{L^{2^*_s}(\mathbb{R}^N)} \leqslant C\,\|\psi-\psi_j\|_{L^\beta(\mathbb{R}^N)} \to 0,$$

and

$$[J\psi - J\psi_j]_{\dot{H}^s(\mathbb{R}^N)} = [J(\psi-\psi_j)]_{\dot{H}^s(\mathbb{R}^N)} \leqslant C\,\|\psi-\psi_j\|_{L^\beta(\mathbb{R}^N)} \to 0,$$

as $j \to +\infty$. Therefore, setting $\Psi_j := J\psi_j$, the last two formulas say that

$$\Psi_j \text{ converges to } \Psi \text{ in } L^{2^*_s}(\mathbb{R}^N) \text{ and in } \dot{H}^s(\mathbb{R}^N) \text{ as } j \to +\infty. \tag{4.1.28}$$

Moreover, we observe that, by (4.1.23), there exists a sequence of smooth and rapidly decreasing functions $\Psi_{j,k}$ such that $\Psi_{j,k}$ converges to Ψ_j in $H^1(\mathbb{R}^N)$ as $k \to +\infty$, and so $\Psi_{j,k}$ converges to Ψ_j in $H^s(\mathbb{R}^N)$ as $k \to +\infty$, thanks to (4.1.27). By the Sobolev immersion (see [82, Theorem 6.5]), we have that $\Psi_{j,k}$ converges to Ψ_j in $L^{2^*_s}(\mathbb{R}^N)$ as $k \to +\infty$. Hence, using also (4.1.28) we obtain that $\Psi = J\psi \in \dot{H}^s(\mathbb{R}^N)$, and this concludes the proof of (4.1.13).

Now we prove (4.1.14). For this, we use (4.1.13) to see that

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} \frac{\big|(\Psi_j - \Psi)(x) - (\Psi_j - \Psi)(y)\big|^2}{|x-y|^{N+2s}}\, dx\, dy = [\Psi_j - \Psi]^2_{\dot{H}^s(\mathbb{R}^N)} \\ = [J(\psi_j - \psi)]^2_{\dot{H}^s(\mathbb{R}^N)} \leqslant C\, \|\psi - \psi_j\|^2_{L^\beta(\mathbb{R}^N)} \to 0\end{aligned}$$

as $j \to +\infty$. This says that the sequence of functions

$$M_j(x,y) := \frac{\Psi_j(x) - \Psi_j(y)}{|x-y|^{\frac{N+2s}{2}}},$$

converges in $L^2(\mathbb{R}^{2N})$ to the function

$$M(x,y) := \frac{\Psi(x) - \Psi(y)}{|x-y|^{\frac{N+2s}{2}}}.$$

In particular, this implies weak convergence in $L^2(\mathbb{R}^{2N})$, that is,

$$\lim_{j\to+\infty} \iint_{\mathbb{R}^{2N}} M_j(x,y)\, \gamma(x,y)\, dx\, dy = \iint_{\mathbb{R}^{2N}} M(x,y)\, \gamma(x,y)\, dx\, dy,$$

for any $\gamma \in L^2(\mathbb{R}^{2N})$.

Thus, if ϕ is smooth and rapidly decreasing, we can take

$$\gamma(x,y) := \frac{\phi(x) - \phi(y)}{|x-y|^{\frac{N+2s}{2}}},$$

and obtain that

$$\begin{aligned}\lim_{j\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{\big(\Psi_j(x) - \Psi_j(y)\big)\big(\phi(x) - \phi(y)\big)}{|x-y|^{N+2s}}\, dx\, dy \\ = \iint_{\mathbb{R}^{2N}} \frac{\big(\Psi(x) - \Psi(y)\big)\big(\phi(x) - \phi(y)\big)}{|x-y|^{N+2s}}\, dx\, dy.\end{aligned}$$

Moreover, since ψ_j converges to ψ in $L^\beta(\mathbb{R}^N)$, we have that

$$\lim_{j\to+\infty} \int_{\mathbb{R}^N} \psi_j(x)\, \phi(x)\, dx = \int_{\mathbb{R}^N} \psi(x)\, \phi(x)\, dx.$$

Consequently, we can pass to the limit in (4.1.25) and obtain (4.1.14) for any ϕ which is smooth and rapidly decreasing.

It remains to establish (4.1.14) for any $\phi \in X^s$. For this, we fix $\phi \in X^s$ and we take a sequence ϕ_k of smooth and rapidly decreasing functions that converge to ϕ in $\dot{H}^s(\mathbb{R}^N)$, and so, by Lemma 4.1.3, also in $L^{2^*_s}(\mathbb{R}^N)$. Moreover, we know that $\Psi \in \dot{H}^s(\mathbb{R}^N)$, thanks to (4.1.13). In particular, by Cauchy-Schwarz and Hölder inequalities, we obtain that

$$\begin{aligned} &\left| \iint_{\mathbb{R}^{2N}} \frac{\big(\Psi(x)-\Psi(y)\big)\big((\phi-\phi_k)(x)-(\phi-\phi_k)(y)\big)}{|x-y|^{N+2s}}\,dx\,dy \right| \\ &\qquad \leqslant [\Psi]_{\dot{H}^s(\mathbb{R}^N)}\,[\phi-\phi_k]_{\dot{H}^s(\mathbb{R}^N)} \to 0, \\ \text{and } &\left| \int_{\mathbb{R}^N} \psi(x)\Big(\phi(x)-\phi_k(x)\Big)\,dx \right| \leqslant \|\psi\|_{L^\beta(\mathbb{R}^N)}\,\|\phi-\phi_k\|_{L^{2^*_s}(\mathbb{R}^N)} \to 0, \end{aligned}$$

as $k \to +\infty$. Therefore, we can write (4.1.14) for the smooth and rapidly decreasing functions ϕ_k, pass to the limit in k, and so obtain (4.1.14) for $\phi \in X^s$. This completes the proof of (4.1.14).

Now we prove (4.1.15). For this, we use the Hölder inequality with exponents β and $2N/(N-2s)$ to compute

$$\begin{aligned} |J\psi(x)| &\leqslant \int_{\mathbb{R}^N} \frac{|\psi(x-y)|}{|y|^{N-2s}}\,dy \\ &\leqslant \int_{B_1} \frac{\|\psi\|_{L^\infty(B_1)}}{|y|^{N-2s}}\,dy + \int_{\mathbb{R}^N\setminus B_1} \frac{|\psi(x-y)|}{|y|^{N-2s}}\,dy \\ &\leqslant C\,\|\psi\|_{L^\infty(B_1)} + \left[\int_{\mathbb{R}^N\setminus B_1} |\psi(x-y)|^\beta\,dy\right]^{\frac{1}{\beta}} \left[\int_{\mathbb{R}^N\setminus B_1} \frac{dy}{|y|^{2N}}\,dy\right]^{\frac{N-2s}{2N}} \\ &\leqslant C\left(\|\psi\|_{L^\infty(\mathbb{R}^N)} + \|\psi\|_{L^\beta(\mathbb{R}^N)}\right), \end{aligned}$$

and this establishes (4.1.15). □

We show now the L^∞ boundedness of the solutions to a general subcritical and critical problem.

Proposition 4.1.6. *Let $u \in \dot{H}^s(\mathbb{R}^N)$ be a positive solution to the problem*

$$(-\Delta)^s u = f(x,u) \qquad \text{in } \mathbb{R}^N,$$

and assume that $|f(x,t)| \leqslant C(1+|t|^p)$, for some $1 \leqslant p \leqslant 2^*_s - 1$ and $C > 0$. Then $u \in L^\infty(\mathbb{R}^N)$.

For the proof of this result we strongly follow the strategy of [32, Proposition 2.2], where this boundedness is proved for the case of bounded domains.

Proof. Let $\beta \geqslant 1$ and $T > 0$, and let us define

$$\varphi(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ t^\beta, & \text{if } 0 < t < T, \\ \beta T^{\beta-1}(t-T) + T^\beta, & \text{if } t \geqslant T. \end{cases}$$

Since φ is convex and Lipschitz, $\varphi(u) \in \dot{H}^s(\mathbb{R}^N)$ and

$$(-\Delta)^s\varphi(u) \leqslant \varphi'(u)(-\Delta)^s u$$

in the weak sense. Moreover, $\varphi(u)\varphi'(u) \in \dot{H}^s(\mathbb{R}^N)$, and hence

$$\begin{aligned}
[\varphi(u)]^2_{\dot{H}^s(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} (-\Delta)^{s/2}[\varphi(u)\varphi'(u)](-\Delta)^{s/2}u\,dx \\
&\leqslant C\int_{\mathbb{R}^N} \varphi(u)\varphi'(u)(1+u^{2_s^*-1})\,dx \\
&= C\left(\int_{\mathbb{R}^N} \varphi(u)\varphi'(u)\,dx + \int_{\mathbb{R}^N} \varphi(u)\varphi'(u)u^{2_s^*-1}\,dx\right).
\end{aligned}$$

Using that $\varphi(u)\varphi'(u) \leqslant \beta u^{2\beta-1}$ and $u\varphi'(u) \leqslant \beta\varphi(u)$, and the Sobolev embedding, we obtain

$$\left(\int_{\mathbb{R}^N} (\varphi(u))^{2_s^*}\right)^{2/2_s^*} \leqslant C\beta\left(\int_{\mathbb{R}^N} u^{2\beta-1}\,dx + \int_{\mathbb{R}^N} (\varphi(u))^2 u^{2^*-2}\,dx\right), \tag{4.1.29}$$

where C is a positive constant that does not depend on β. Notice that the last integral is well defined for every T in the definition of φ. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^N} (\varphi(u))^2 u^{2^*-2}\,dx &= \int_{\{u\leqslant T\}} (\varphi(u))^2 u^{2_s^*-2}\,dx + \int_{\{u>T\}} (\varphi(u))^2 u^{2^*-2}\,dx \\
&\leqslant T^{2\beta-2}\int_{\mathbb{R}^N} u^{2_s^*}\,dx + C\int_{\mathbb{R}^N} u^{2_s^*}\,dx < +\infty,
\end{aligned}$$

where we have used that $\beta > 1$ and that $\varphi(u)$ is linear where $u \geqslant T$. We choose now β in (4.1.29) such that $2\beta - 1 = 2_s^*$, and we name it β_1, that is,

$$\beta_1 = \frac{2_s^*+1}{2}.$$

Let $R > 0$ to be fixed later. Attending to the last integral in (4.1.29) and applying Hölder's inequality with exponents $r = 2_s^*/2$ and $r' = 2_s^*/(2_s^*-2)$,

(4.1.30)

$$\begin{aligned}
\int_{\mathbb{R}^N} (\varphi(u))^2 u^{2_s^*-2}\,dx &= \int_{\{u\leqslant R\}} (\varphi(u))^2 u^{2_s^*-2}\,dx + \int_{\{u>R\}} (\varphi(u))^2 u^{2_s^*-2}\,dx \\
&\leqslant \int_{\{u\leqslant R\}} \frac{(\varphi(u))^2}{u} R^{2_s^*-1}\,dx + \left(\int_{\mathbb{R}^N} (\varphi(u))^{2_s^*}\,dx\right)^{2/2_s^*}\left(\int_{\{u>R\}} u^{2_s^*}\,dx\right)^{\frac{2_s^*-2}{2_s^*}}.
\end{aligned}$$

By the Monotone Convergence Theorem, we can choose R large enough so that

$$\left(\int_{\{u>R\}} u^{2_s^*}\,dx\right)^{\frac{2_s^*-2}{2_s^*}} \leqslant \frac{1}{2C\beta_1},$$

where C is the constant appearing in (4.1.29). Therefore, we can absorb the last term in (4.1.30) by the left hand side of (4.1.29) to get

$$\left(\int_{\mathbb{R}^N} (\varphi(u))^{2_s^*}\, dx\right)^{2/2_s^*} \leqslant 2C\beta_1 \left(\int_{\mathbb{R}^N} u^{2_s^*}\, dx + R^{2_s^*-1} \int_{\mathbb{R}^N} \frac{(\varphi(u))^2}{u}\, dx\right).$$

Making $T \to \infty$ in the definition of φ, this inequality becomes

$$\left(\int_{\mathbb{R}^N} u^{2_s^*\beta_1}\, dx\right)^{2/2_s^*} \leqslant 2C\beta_1 \left(\int_{\mathbb{R}^N} u^{2_s^*}\, dx + R^{2_s^*-1} \int_{\mathbb{R}^N} u^{2_s^*}\, dx\right) < +\infty,$$

and therefore $u \in L^{2_s^*\beta_1}(\mathbb{R}^N)$.

Let us suppose now $\beta > \beta_1$. Thus, using that $\varphi(u) \leqslant u^\beta$ in the right hand side of (4.1.29) and letting $T \to \infty$ we get

$$\left(\int_{\mathbb{R}^N} u^{2_s^*\beta}\, dx\right)^{2/2_s^*} \leqslant C\beta \left(\int_{\mathbb{R}^N} u^{2\beta-1}\, dx + \int_{\mathbb{R}^N} u^{2\beta+2_s^*-2}\, dx\right). \tag{4.1.31}$$

Furthermore, we can write

$$u^{2\beta-1} = u^a u^b,$$

with $a := \frac{2_s^*(2_s^*-1)}{2(\beta-1)}$, $b := 2\beta - 1 - a$. Notice that, since $\beta > \beta_1$, then $0 < a, b < 2_s^*$. Hence, applying Young's inequality with exponents

$$r = 2_s^*/a \text{ and } r' = 2^* - s/(2_s^* - a),$$

there holds

$$\begin{aligned}
\int_{\mathbb{R}^N} u^{2\beta-1}\, dx &\leqslant \frac{a}{2_s^*} \int_{\mathbb{R}^N} u^{2_s^*}\, dx + \frac{2_s^* - a}{2_s^*} \int_{\mathbb{R}^N} u^{\frac{2_s^* b}{2_s^* - a}}\, dx \\
&\leqslant \int_{\mathbb{R}^N} u^{2_s^*}\, dx + \int_{\mathbb{R}^N} u^{2\beta+2_s^*-2}\, dx \\
&\leqslant C\left(1 + \int_{\mathbb{R}^N} u^{2\beta+2_s^*-2}\, dx\right),
\end{aligned}$$

with $C > 0$ independent of β. Plugging this into (4.1.31),

$$\left(\int_{\mathbb{R}^N} u^{2_s^*\beta}\, dx\right)^{2/2_s^*} \leqslant C\beta \left(1 + \int_{\mathbb{R}^N} u^{2\beta+2_s^*-2}\, dx\right),$$

with C changing from line to line, but remaining independent of β. Therefore,

$$\left(1 + \int_{\mathbb{R}^N} u^{2_s^*\beta}\, dx\right)^{\frac{1}{2_s^*(\beta-1)}} \leqslant (C\beta)^{\frac{1}{2(\beta-1)}} \left(1 + \int_{\mathbb{R}^N} u^{2\beta+2_s^*-2}\, dx\right)^{\frac{1}{2(\beta-1)}}, \tag{4.1.32}$$

that is (2.6) in [32, Proposition 2.2]. From now on, we follow exactly their iterarive argument. That is, we define β_{m+1}, $m \geqslant 1$, so that

$$2\beta_{m+1} + 2_s^* - 2 = 2_s^*\beta_m.$$

Thus,

$$\beta_{m+1} - 1 = \left(\frac{2_s^*}{2}\right)^m (\beta_1 - 1),$$

and replacing in (4.1.32) it yields

$$\left(1 + \int_{\mathbb{R}^N} u^{2_s^* \beta_{m+1}} \, dx\right)^{\frac{1}{2_s^*(\beta_{m+1}-1)}} \leqslant (C\beta_{m+1})^{\frac{1}{2(\beta_{m+1}-1)}} \left(1 + \int_{\mathbb{R}^N} u^{2_s^* \beta_m} \, dx\right)^{\frac{1}{2_s^*(\beta_m-1)}}.$$

Defining $C_{m+1} := C\beta_{m+1}$ and

$$A_m := \left(1 + \int_{\mathbb{R}^N} u^{2_s^* \beta_m} \, dx\right)^{\frac{1}{2_s^*(\beta_m-1)}},$$

we conclude that there exists a constant $C_0 > 0$ such that

$$A_{m+1} \leqslant \prod_{k=2}^{m+1} C_k^{\frac{1}{2(\beta_k-1)}} A_1 \leqslant C_0 A_1.$$

Thus,

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant C_0 A_1 < +\infty,$$

since we already proved that $u \in L^{2_s^* \beta_1}(\mathbb{R}^N)$. □

4.2 The Lyapunov-Schmidt reduction.

In this section we perform the so-called Lyapunov-Schmidt reduction. Since the argument is delicate and involves many lemmata, we will perform it in different steps.

4.2.1 Functional setting.

Given $0 < \mu_1 < \mu_2$ and $R > 0$, we define the manifold

$$Z_0 := \{z_{\mu,\xi} \text{ s.t. } \mu_1 < \mu < \mu_2, \ |\xi| < R\}, \tag{4.2.1}$$

where $z_{\mu,\xi}$ was introduced in (4.1.3). We will make our choice of R, μ_1 and μ_2 later on. Notice that the functions in Z_0 are critical points of $\mathcal{I}_0$, as defined in (4.1.6).

We will often implicitly identify Z_0 with the subdomain $(\mu_1, \mu_2) \times B_R$ of $\mathbb{R}^{N+1}$ described by coordinates (μ, ξ).

In order to apply the abstract variational method discussed in the introduction, we would need in principle the functional $\mathcal{I}_\varepsilon$ defined in (4.1.4) to be C^2 on $\dot{H}^s(\mathbb{R}^N)$. Unfortunately, this is not true if $q < 1$, and therefore, in order to treat the whole set of values $q \in (0, p)$, we recall that ω is the support of the function h and we set

$$\begin{aligned} a &:= \inf\{z_{\mu,\xi}(x) \text{ s.t. } x \in \omega, \ \mu_1 < \mu < \mu_2, \ |\xi| < R\}, \\ V &:= \{w \in X^s \text{ s.t } \|w\|_{X^s} < a/2\}, \\ \text{and } U &:= \{u := z_{\mu,\xi} + w \text{ s.t. } z_{\mu,\xi} \in Z_0, \ w \in V\}. \end{aligned} \tag{4.2.2}$$

We observe that, if $u \in U$ and $x \in \omega$, then

$$u(x) = z_{\mu,\xi}(x) + w(x) \geqslant a - \|w\|_{L^\infty(\mathbb{R}^N)} \geqslant a - \|w\|_{X^s} > a - \frac{a}{2} = \frac{a}{2},$$

and so

$$u(x) > \frac{a}{2} > 0 \quad \text{for any } x \in \omega. \tag{4.2.3}$$

Therefore, recalling (4.1.7), we obtain that the functional $\mathcal{I}$ is C^2 on U. Hence, also $\mathcal{I}_\varepsilon : U \to \mathbb{R}$ is of class C^2.

Now, we set

$$q_j := \frac{\partial z_{\mu,\xi}}{\partial \xi_j}, \; j = 1, \dots, N, \quad \text{and} \quad q_{N+1} := \frac{\partial z_{\mu,\xi}}{\partial \mu}, \tag{4.2.4}$$

and we notice that q_j satisfies

$$(-\Delta)^s q_j = p z_{\mu,\xi}^{p-1} q_j \quad \text{in } \mathbb{R}^N \tag{4.2.5}$$

for every $j = 1, \dots, N+1$. We also denote by

$$T_{z_{\mu,\xi}} Z_0 := span\,\{q_1, \dots, q_{N+1}\}$$

the tangent space to Z_0 at $z_{\mu,\xi}$.

Moreover, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\dot{H}^s(\mathbb{R}^N)$, that is, for any $v_1, v_2 \in \dot{H}^s(\mathbb{R}^N)$,

$$\langle v_1, v_2 \rangle = \iint_{\mathbb{R}^{2N}} \frac{\big(v_1(x) - v_1(y)\big)\big(v_2(x) - v_2(y)\big)}{|x-y|^{N+2s}}\, dx\, dy.$$

We also define the notion of orthogonality with respect to such scalar product and we denote it by $\perp$. That is, we set

$$\big(T_{z_{\mu,\xi}} Z_0\big)^\perp := \Big\{ v \in \dot{H}^s(\mathbb{R}^N) \text{ s.t. } \langle v, \phi \rangle = 0 \text{ for all } \phi \in T_{z_{\mu,\xi}} Z_0 \Big\}.$$

In particular, we prove the following orthogonality result.

Lemma 4.2.1. *There exist $\lambda_i = \lambda_i(\mu, \xi)$, for $i = 1, \dots, N+1$, such that*

$$\langle q_i, q_j \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ \lambda_i & \text{if } i = j, \end{cases}$$

and

$$\inf_{\substack{\mu \in (\mu_1, \mu_2) \\ |\xi| < R \\ i \in \{1, \dots, N+1\}}} \lambda_i(\mu, \xi) > 0.$$

Proof. For any $r \geqslant 0$, we write

$$\bar{z}(r) := \frac{\alpha_{N,s}}{(1+r)^{(N-2s)/2}}.$$

In this way $z_0(x) = \bar{z}(|x|^2)$ and so

$$z_{\mu,\xi}(x) = \mu^{(2s-N)/2}\bar{z}\left(\frac{|x-\xi|^2}{\mu^2}\right).$$

So we obtain that

$$\frac{\partial z_{\mu,\xi}}{\partial \xi_i}(x) = \mu^{(2s-N)/2}\bar{z}'\left(\frac{|x-\xi|^2}{\mu^2}\right)\frac{2(\xi_i - x_i)}{\mu^2},$$

and therefore

$$\frac{\partial z_{\mu,\xi}}{\partial \xi_i}(y+\xi) = \mu^{(2s-N)/2}\bar{z}'\left(\frac{|y|^2}{\mu^2}\right)\frac{2(-y_i)}{\mu^2},$$

which is odd in the variable y_i.

Similarly,

$$\frac{\partial z_{\mu,\xi}}{\partial \mu}(x) = \frac{2s-N}{2}\mu^{(2s-N-2)/2}\bar{z}\left(\frac{|x-\xi|^2}{\mu^2}\right) - \mu^{(2s-N)/2}\bar{z}'\left(\frac{|x-\xi|^2}{\mu^2}\right)\frac{2|x-\xi|^2}{\mu^3},$$

and thus

$$\frac{\partial z_{\mu,\xi}}{\partial \mu}(y+\xi) = \frac{2s-N}{2}\mu^{(2s-N-2)/2}\bar{z}\left(\frac{|y|^2}{\mu^2}\right) - \mu^{(2s-N)/2}\bar{z}'\left(\frac{|y|^2}{\mu^2}\right)\frac{2|y|^2}{\mu^3}, \tag{4.2.6}$$

that is even in any of the variables y_i.

Notice also that

$$z_{\mu,\xi}(y+\xi) = \mu^{(2s-N)/2}\bar{z}\left(\frac{|y|^2}{\mu^2}\right),$$

which is also even in any of the variables y_i. As a consequence, using the change of variable $x = y+\xi$ we obtain that, for any i, $j \in \{1,\dots,N\}$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} z_{\mu,\xi}^{p-1}(x)\,\frac{\partial z_{\mu,\xi}}{\partial \xi_i}(x)\,\frac{\partial z_{\mu,\xi}}{\partial \xi_j}(x)\,dx \\
&\qquad = \int_{\mathbb{R}^N} z_{\mu,\xi}^{p-1}(y+\xi)\,\frac{\partial z_{\mu,\xi}}{\partial \xi_i}(y+\xi)\,\frac{\partial z_{\mu,\xi}}{\partial \xi_j}(y+\xi)\,dy \\
&\qquad = \int_{\mathbb{R}^N} \mu^{(2s-N)(p+1)/2}\bar{z}^{p-1}\left(\frac{|y|^2}{\mu^2}\right)(\bar{z}')^2\left(\frac{|y|^2}{\mu^2}\right)\frac{4y_iy_j}{\mu^2}\,dy \\
&\qquad = \begin{cases} 0 & \text{if } i \neq j, \\ c_1 & \text{if } i = j, \end{cases}
\end{aligned} \tag{4.2.7}$$

for some $c_1 > 0$, which is bounded from zero uniformly.

Likewise, for any $i \in \{1, \dots, N\}$,

$$\int_{\mathbb{R}^N} z_{\mu,\xi}^{p-1}(x) \, \frac{\partial z_{\mu,\xi}}{\partial \xi_i}(x) \, \frac{\partial z_{\mu,\xi}}{\partial \mu}(x) \, dx = 0. \tag{4.2.8}$$

Finally, we observe that $\bar{z}$ is positive and decreasing, thus both $\bar{z}$ and $-\bar{z}'$ are positive: this says that the right hand side of (4.2.6) is positive, and indeed bounded from zero uniformly. Hence we obtain that

$$\int_{\mathbb{R}^N} z_{\mu,\xi}^{p-1}(x) \, \left(\frac{\partial z_{\mu,\xi}}{\partial \mu}(x) \right)^2 dx = c_2, \tag{4.2.9}$$

with $c_2 > 0$ and bounded from zero uniformly.

Now, to make the notation uniform, we take ζ, $\eta \in \{\xi_1, \dots, \xi_n, \mu\}$ and we consider the derivatives of $z_{\mu,\xi}$ with respect to ζ and η. Then we have that the quantity

$$\left\langle \frac{\partial z_{\mu,\xi}}{\partial \zeta}, \frac{\partial z_{\mu,\xi}}{\partial \eta} \right\rangle,$$

is equal, up to dimensional constants, to

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta)^{s/2} \frac{\partial z_{\mu,\xi}}{\partial \zeta}(x) \, (-\Delta)^{s/2} \frac{\partial z_{\mu,\xi}}{\partial \eta}(x) \, dx \\
&\qquad = \int_{\mathbb{R}^N} (-\Delta)^s \frac{\partial z_{\mu,\xi}}{\partial \zeta}(x) \, \frac{\partial z_{\mu,\xi}}{\partial \eta}(x) \, dx \\
&\qquad = \int_{\mathbb{R}^N} \frac{\partial}{\partial \zeta} (-\Delta)^s z_{\mu,\xi}(x) \, \frac{\partial z_{\mu,\xi}}{\partial \eta}(x) \, dx \\
&\qquad = \int_{\mathbb{R}^N} \frac{\partial}{\partial \zeta} z_{\mu,\xi}^p(x) \, \frac{\partial z_{\mu,\xi}}{\partial \eta}(x) \, dx \\
&\qquad = p \int_{\mathbb{R}^N} z_{\mu,\xi}^{p-1}(x) \frac{\partial z_{\mu,\xi}}{\partial \zeta}(x) \, \frac{\partial z_{\mu,\xi}}{\partial \eta}(x) \, dx,
\end{aligned}$$

hence the desired result follows from (4.2.7), (4.2.8) and (4.2.9). □

Concerning the statement of Lemma 4.2.1, we point out that the proof shows that

$$\lambda_1 = \dots = \lambda_N$$

(while λ_{N+1} could be different), but in this work we are not taking advantage of this additional feature.

4.2.2 Solving an auxiliary equation.

Keeping the notation introduced in the previous subsection, the goal now is to solve an auxiliary equation by means of the Implicit Function Theorem to obtain the following result.

Lemma 4.2.2. *Let $z_{\mu,\xi} \in Z_0$. Then, for $\varepsilon > 0$ sufficiently small, there exists a unique $w = w(\varepsilon, z_{\mu,\xi}) \in \big(T_{z_{\mu,\xi}} Z_0\big)^{\perp}$ such that*

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{\big((z_{\mu,\xi}+w)(x) - (z_{\mu,\xi}+w)(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s}}\,dx\,dy \\ &\qquad = \int_{\mathbb{R}^N} \Big(\varepsilon h(x)\big(z_{\mu,\xi}(x)+w(x)\big)^q + \big(z_{\mu,\xi}(x)+w(x)\big)^p\Big)\,\varphi(x)\,dx,\end{aligned} \tag{4.2.10}$$

for any $\varphi \in \big(T_{z_{\mu,\xi}} Z_0\big)^{\perp} \cap X^s$.

Moreover, the function w is of class C^1 with respect to μ and ξ and there exists a constant $C > 0$ such that

$$\|w\|_{X^s} \leqslant C\,\varepsilon, \quad \textit{and} \quad \lim_{\varepsilon\to 0} \left\|\frac{\partial w}{\partial \mu}\right\|_{X^s} + \left\|\frac{\partial w}{\partial \xi}\right\|_{X^s} = 0. \tag{4.2.11}$$

Indeed, recalling the definition of U given in (4.2.2), we can set for any $u \in U$

$$A_\varepsilon(u) := \varepsilon\, h\, u^q + u^p. \tag{4.2.12}$$

We observe that $u = J(A_\varepsilon(u))$ (where J has been introduced in (4.1.11)) implies that u solves (up to an unessential renormalizing constant that we neglect for simplicity, recall the Remark 4.1.5)

$$(-\Delta)^s u = A_\varepsilon(u) \ \text{ in } \mathbb{R}^N,$$

thanks to Theorem 4.1.4 (see in particular (4.1.14)). Moreover, we have that

$$\|J(A_\varepsilon(u))\|_{L^{2^*_s}(\mathbb{R}^N)} < +\infty. \tag{4.2.13}$$

Indeed, by (4.1.12) in Theorem 4.1.4 we get that there exists $C > 0$ such that

$$\|J(A_\varepsilon(u))\|_{L^{2^*_s}(\mathbb{R}^N)} \leqslant C\|A_\varepsilon(u)\|_{L^\beta(\mathbb{R}^N)}, \tag{4.2.14}$$

where $\beta := 2N/(N+2s)$. Now, since $u \in L^{2^*_s}(\mathbb{R}^N)$ and $p = (N+2s)/(N-2s)$, we have that $u^p \in L^\beta(\mathbb{R}^N)$. This and the fact that h is continuous and compactly supported imply that $\|A_\varepsilon(u)\|_{L^\beta(\mathbb{R}^N)} < +\infty$. Therefore, from (4.2.14) we deduce (4.2.13).

Analogously, since $u \in L^\infty(\mathbb{R}^N)$, making use of (4.1.13) and (4.1.15), one sees that

$$[J(A_\varepsilon(u))]_{\dot{H}^s(\mathbb{R}^N)} + \|J(A_\varepsilon(u))\|_{L^\infty(\mathbb{R}^N)} < +\infty.$$

Hence, using Theorem 4.1.4, we have that if $u \in U$ then $J(A_\varepsilon(u)) \in X^s$.

Now, we use the notation $U \ni u := z_{\mu,\xi} + w$, with $z_{\mu,\xi} \in Z_0$ and $w \in V$, and we recall that we are identifying the manifold Z_0 defined in (4.2.1) with $(\mu_1,\mu_2) \times B_R \subset \mathbb{R}^{N+1}$. We define

$$H : (\mu_1,\mu_2) \times B_R \times \mathbb{R} \times V \times \mathbb{R}^{N+1} \to X^s \times \mathbb{R}^{N+1} \tag{4.2.15}$$

as $H = (H_1, H_2)$, with components

$$\begin{aligned} H_1(\mu,\xi,\varepsilon,w,\alpha) &:= z_{\mu,\xi} + w - J(A_\varepsilon(z_{\mu,\xi}+w)) - \sum_{i=1}^{N+1} \alpha_i\, q_i, \\ H_2(\mu,\xi,\varepsilon,w,\alpha) &:= (\langle w, q_1\rangle, \dots, \langle w, q_{N+1}\rangle)\,, \end{aligned}$$

where q_i was defined in (4.2.4).

Our goal is to find $w = w(\varepsilon, z_{\mu,\xi})$ (that we also think as $w = w(\varepsilon,\mu,\xi)$ with a slight abuse of notation) that solves the equation $H(\mu,\xi,\varepsilon,w,\alpha) = 0$, that is the system of equations

$$H_1(\mu,\xi,\varepsilon,w,\alpha) = 0 = H_2(\mu,\xi,\varepsilon,w,\alpha). \tag{4.2.16}$$

We notice that if w satisfies (4.2.16) then $w \in \big(T_{z_{\mu,\xi}} Z_0\big)^\perp$ and $z_{\mu,\xi} + w$ is a solution of the auxiliary equation (4.2.10). Indeed, $H_2(\mu,\xi,\varepsilon,w,\alpha) = 0$ implies that

$$\langle w, q_i\rangle = 0 \quad \text{for any } i = 1,\dots,N+1,$$

which means that $w \in \big(T_{z_{\mu,\xi}} Z_0\big)^\perp$. Moreover, $H_1(\mu,\xi,\varepsilon,w,\alpha) = 0$ gives that

$$z_{\mu,\xi} + w - J(A_\varepsilon(z_{\mu,\xi}+w)) \in T_{z_{\mu,\xi}} Z_0,$$

and so

$$\langle z_{\mu,\xi} + w - J(A_\varepsilon(z_{\mu,\xi}+w)), \varphi\rangle = 0$$

for any $\varphi \in \big(T_{z_{\mu,\xi}} Z_0\big)^\perp \cap X^s$. That is

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{\big((z_{\mu,\xi}+w)(x) - (z_{\mu,\xi}+w)(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s}}\,dx\,dy \\ &= \iint_{\mathbb{R}^{2N}} \frac{\big(J(A_\varepsilon(z_{\mu,\xi}+w))(x) - J(A_\varepsilon(z_{\mu,\xi}+w))(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s}}\,dx\,dy \\ &= \int_{\mathbb{R}^N} A_\varepsilon(z_{\mu,\xi}+w)(x)\,\varphi(x)\,dx, \end{aligned} \tag{4.2.17}$$

for any $\varphi \in \big(T_{z_{\mu,\xi}} Z_0\big)^\perp \cap X^s$, thanks to (4.1.14) in Theorem 4.1.4, which is (4.2.10).

Therefore, to prove Lemma 4.2.2, the strategy will be to apply the Implicit Function Theorem to find a solution of the auxiliary equation $H(\mu,\xi,\varepsilon,w,\alpha) = 0$. Since we are working in the space X^s, it is not obvious that H satisfies the hypotheses needed to apply this theorem. Indeed, the proofs of these requirements are very technically involved, so we devote the next two subsections to study in detail the behavior of the operator H.

Preliminary results on H.

Consider the operator defined in (4.2.15). First of all, we prove some continuity property.

Lemma 4.2.3. *H is C^1 with respect to w.*

Proof. We first notice that H_2 depends linearly on w, and so it is C^1. Now we prove that H_1 is continuous in X^s. Indeed, for any $w_1, w_2 \in V$ we have that

$$H_1(\mu,\xi,\varepsilon,w_1,\alpha) - H_1(\mu,\xi,\varepsilon,w_2,\alpha) = w_1 - w_2 - J(A_\varepsilon(z_{\mu,\xi}+w_1)) + J(A_\varepsilon(z_{\mu,\xi}+w_1)),$$

and therefore

$$\begin{aligned} &\|H_1(\mu,\xi,\varepsilon,w_1,\alpha) - H_1(\mu,\xi,\varepsilon,w_2,\alpha)\|_{X^s} \\ &\qquad \leqslant \|w_1-w_2\|_{X^s} + \|J(A_\varepsilon(z_{\mu,\xi}+w_1)) - J(A_\varepsilon(z_{\mu,\xi}+w_2))\|_{X^s}. \end{aligned} \tag{4.2.18}$$

By (4.1.13) and (4.1.15) of Theorem 4.1.4 and the fact that J is linear we deduce that

$$\begin{aligned} &\|J(A_\varepsilon(z_{\mu,\xi}+w_1)) - J(A_\varepsilon(z_{\mu,\xi}+w_2))\|_{X^s} \\ &\leqslant C\left(\|A_\varepsilon(z_{\mu,\xi}+w_1) - A_\varepsilon(z_{\mu,\xi}+w_2)\|_{L^\infty(\mathbb{R}^N)} + \|A_\varepsilon(z_{\mu,\xi}+w_1) - A_\varepsilon(z_{\mu,\xi}+w_2)\|_{L^\beta(\mathbb{R}^N)}\right), \end{aligned} \tag{4.2.19}$$

where $\beta := 2N/(N+2s)$. Now from (4.2.12) we deduce that

$$\begin{aligned} &A_\varepsilon(z_{\mu,\xi}+w_1) - A_\varepsilon(z_{\mu,\xi}+w_2) \\ &\qquad = \varepsilon\, h\, [(z_{\mu,\xi}+w_1)^q - (z_{\mu,\xi}+w_2)^q] + (z_{\mu,\xi}+w_1)^p - (z_{\mu,\xi}+w_2)^p \\ &\qquad = \varepsilon q\, h\,(z_{\mu,\xi}+\tilde{w})^{q-1}(w_1-w_2) + p(z_{\mu,\xi}+\bar{w})^{p-1}(w_1-w_2), \end{aligned}$$

for some $\tilde{w}, \bar{w}$ on the segment joining w_1 and w_2 (in particular $\tilde{w}, \bar{w} \in L^{2^*_s}(\mathbb{R}^N)$ and $z_{\mu,\xi}+\tilde{w}, z_{\mu,\xi}+\bar{w}$ satisfy (4.2.3)). Consequently,

$$\|A_\varepsilon(z_{\mu,\xi}+w_1) - A_\varepsilon(z_{\mu,\xi}+w_2)\|_{L^\infty(\mathbb{R}^N)} \leqslant C\,\|w_1-w_2\|_{L^\infty(\mathbb{R}^N)}. \tag{4.2.20}$$

Moreover, since h has compact support, we have that

$$\|\varepsilon\, h\,(z_{\mu,\xi}+\tilde{w})^{q-1}(w_1-w_2)\|_{L^\beta(\mathbb{R}^N)} \leqslant C\,\|w_1-w_2\|_{L^\infty(\mathbb{R}^N)}. \tag{4.2.21}$$

Finally, using Hölder inequality with exponents

$$2^*_s/\beta = (N+2s)/(N-2s) \text{ and } \delta := (N+2s)/4s,$$

we get

$$\begin{aligned} &\|(z_{\mu,\xi}+\bar{w})^{p-1}(w_1-w_2)\|^\beta_{L^\beta(\mathbb{R}^N)} \\ &\qquad = \int_{\mathbb{R}^N} (z_{\mu,\xi}+\bar{w})^{(p-1)\beta}(w_1-w_2)^\beta\,dx \\ &\qquad \leqslant \left(\int_{\mathbb{R}^N} (z_{\mu,\xi}+\bar{w})^{(p-1)\beta\delta}\,dx\right)^{1/\delta} \left(\int_{\mathbb{R}^N} (w_1-w_2)^{2^*_s}\,dx\right)^{\beta/2^*_s} \\ &\qquad = \left(\int_{\mathbb{R}^N} (z_{\mu,\xi}+\bar{w})^{2^*_s}\,dx\right)^{1/\delta} \left(\int_{\mathbb{R}^N} (w_1-w_2)^{2^*_s}\,dx\right)^{\beta/2^*_s} \\ &\qquad \leqslant C\,\|w_1-w_2\|^\beta_{L^{2^*_s}(\mathbb{R}^N)} \\ &\qquad \leqslant C\,[w_1-w_2]^\beta_{\dot{H}^s(\mathbb{R}^N)}, \end{aligned}$$

up to renaming $C>0$, where we have used Lemma 4.1.3 in the last line. Using this, (4.2.20) and (4.2.21) into (4.2.19) we obtain that

$$\|J(A_\varepsilon(z_{\mu,\xi}+w_1))-J(A_\varepsilon(z_{\mu,\xi}+w_2))\|_{X^s}\leqslant C\,\|w_1-w_2\|_{X^s},$$

which together with (4.2.18) imply that

$$\|H_1(\mu,\xi,\varepsilon,w_1,\alpha)-H_1(\mu,\xi,\varepsilon,w_2,\alpha)\|_{X^s}\leqslant C\,\|w_1-w_2\|_{X^s},$$

up to renaming C. This shows the continuity of H_1 in X^s with respect to w.

Now, in order to prove that H_1 is C^1 with respect to w, we observe that

$$\begin{aligned}\frac{\partial H_1}{\partial w}[v]&=v-J(A'_\varepsilon(z_{\mu,\xi}+w)v)\\&=v-J\left(q\varepsilon\,h\,(z_{\mu,\xi}+w)^{q-1}v+p(z_{\mu,\xi}+w)^{p-1}v\right).\end{aligned}\tag{4.2.22}$$

To see this, we take $v\in V$ and $|t|<1$ and we compute

$$\begin{aligned}&A_\varepsilon(z_{\mu,\xi}+w+tv)-A_\varepsilon(z_{\mu,\xi}+w)\\&\qquad=\varepsilon\,h\,\left[(z_{\mu,\xi}+w+tv)^q-(z_{\mu,\xi}+w)^q\right]+(z_{\mu,\xi}+w+tv)^p-(z_{\mu,\xi}+w)^p\\&\qquad=q\varepsilon\,h\,(z_{\mu,\xi}+w)^{q-1}tv+p(z_{\mu,\xi}+w)^{p-1}tv+O(t^2),\end{aligned}$$

and so

$$\lim_{t\to0}\frac{A_\varepsilon(z_{\mu,\xi}+w+tv)-A_\varepsilon(z_{\mu,\xi}+w)}{t}=q\varepsilon\,h\,(z_{\mu,\xi}+w)^{q-1}v+p(z_{\mu,\xi}+w)^{p-1}v.$$

From this and the fact that J is linear we get that

$$\begin{aligned}\frac{\partial H_1}{\partial w}[v]&=\lim_{t\to0}\frac{1}{t}\left[tv+J\left(A_\varepsilon(z_{\mu,\xi}+w+tv)-A_\varepsilon(z_{\mu,\xi}+w)\right)\right]\\&=v-J\left(q\varepsilon\,h\,(z_{\mu,\xi}+w)^{q-1}v+p(z_{\mu,\xi}+w)^{p-1}v\right),\end{aligned}$$

which is (4.2.22). From (4.2.22) we obtain that, for any $w_1,w_2\in V$,

$$\begin{aligned}&\left\|\frac{\partial H_1}{\partial w}(\mu,\xi,w_1,\varepsilon,\alpha)-\frac{\partial H_1}{\partial w}(\mu,\xi,w_2,\varepsilon,\alpha)\right\|_{\mathcal{L}((X^s)^*,X^s)}\\&\qquad=\sup_{\|v\|_{X^s}=1}\left\|J(A'_\varepsilon(z_{\mu,\xi}+w_1)v)-J(A'_\varepsilon(z_{\mu,\xi}+w_2)v)\right\|_{X^s}.\end{aligned}\tag{4.2.23}$$

Since J is linear, by (4.1.13) and (4.1.15) in Theorem 4.1.4 we obtain that

$$\begin{aligned}&\left\|J(A'_\varepsilon(z_{\mu,\xi}+w_1)v)-J(A'_\varepsilon(z_{\mu,\xi}+w_2)v)\right\|_{X^s}\\&\qquad\leqslant C\Big(\|A'_\varepsilon(z_{\mu,\xi}+w_1)v-A'_\varepsilon(z_{\mu,\xi}+w_2)v\|_{L^\infty(\mathbb{R}^N)}\\&\qquad\qquad+\|A'_\varepsilon(z_{\mu,\xi}+w_1)v-A'_\varepsilon(z_{\mu,\xi}+w_2)v\|_{L^\beta(\mathbb{R}^N)}\Big),\end{aligned}\tag{4.2.24}$$

where $\beta := 2N/(N+2s)$. We have that

$$\begin{aligned}&A'_\varepsilon(z_{\mu,\xi}+w_1)v - A'_\varepsilon(z_{\mu,\xi}+w_2)v\\ &\qquad = q\,\varepsilon\, h\, v\left[(z_{\mu,\xi}+w_1)^{q-1}-(z_{\mu,\xi}+w_2)^{q-1}\right]+p\,v\,\left[(z_{\mu,\xi}+w_1)^{p-1}-(z_{\mu,\xi}+w_2)^{p-1}\right],\end{aligned}$$

and so

$$\begin{aligned}&|A'_\varepsilon(z_{\mu,\xi}+w_1)v - A'_\varepsilon(z_{\mu,\xi}+w_2)v|\\ &\qquad \leqslant q|q-1|\,\varepsilon\,|h|\,|v|\,|z_{\mu,\xi}+\tilde w|^{q-2}|w_1-w_2|+C\,|w_1-w_2|^{p-1}|v|,\end{aligned} \tag{4.2.25}$$

for some $\tilde w$ on the segment joining w_1 and w_2. Accordingly,

$$\begin{aligned}&\|A'_\varepsilon(z_{\mu,\xi}+w_1)v - A'_\varepsilon(z_{\mu,\xi}+w_2)v\|_{L^\infty(\mathbb{R}^N)}\\ &\qquad \leqslant C\,\left(\|w_1-w_2\|_{L^\infty(\mathbb{R}^N)}+\|w_1-w_2\|^{p-1}_{L^\infty(\mathbb{R}^N)}\right),\end{aligned} \tag{4.2.26}$$

since $z_{\mu,\xi}+\tilde w$ satisfies (4.2.3). Concerning the estimate for the L^β-norm, we observe that, since h is compactly supported and $v\in L^\beta_{loc}(\mathbb{R}^N)$, we have

$$\|q|q-1|\,\varepsilon\,|h|\,|v|\,|z_{\mu,\xi}+\tilde w|^{q-2}|w_1-w_2|\|_{L^\beta(\mathbb{R}^N)}\leqslant C\,\|w_1-w_2\|_{L^\infty(\mathbb{R}^N)}. \tag{4.2.27}$$

Moreover, applying Hölder inequality with exponents $\frac{2^*_s}{(p-1)\beta}=\frac{N+2s}{4s}$ and p we obtain that

$$\begin{aligned}\||w_1-w_2|^{p-1}|v|\|^\beta_{L^\beta(\mathbb{R}^N)} &= \int_{\mathbb{R}^N}|w_1-w_2|^{(p-1)\beta}|v|^\beta\,dx\\ &\leqslant \left(\int_{\mathbb{R}^N}|w_1-w_2|^{2^*_s}\,dx\right)^{4s/(N+2s)}\left(\int_{\mathbb{R}^N}|v|^{p\beta}\,dx\right)^{1/p}\\ &= \|w_1-w_2\|^{8Ns/[(N+2s)(N-2s)]}_{L^{2^*_s}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}|v|^{2^*_s}\,dx\right)^{1/p}\\ &\leqslant C\,\|w_1-w_2\|^{8Ns/[(N+2s)(N-2s)]}_{L^{2^*_s}(\mathbb{R}^N)},\end{aligned}$$

for a suitable positive constant C. Hence, by Lemma 4.1.3, we have that

$$\||w_1-w_2|^{p-1}|v|\|_{L^\beta(\mathbb{R}^N)}\leqslant C\,\|w_1-w_2\|^{4s/(N-2s)}_{L^{2^*_s}(\mathbb{R}^N)}\leqslant C\,[w_1-w_2]^{4s/(N-2s)}_{\dot H^s(\mathbb{R}^N)},$$

up to relabelling C. This, (4.2.27) and (4.2.25) imply that

$$\|A'_\varepsilon(z_{\mu,\xi}+w_1)v - A'_\varepsilon(z_{\mu,\xi}+w_2)v\|_{L^\beta(\mathbb{R}^N)}\leqslant C\,\left(\|w_1-w_2\|_{X^s}+\|w_1-w_2\|^{4s/(N-2s)}_{X^s}\right).$$

Putting together this, (4.2.26), (4.2.24) and (4.2.23), we obtain that $\frac{\partial H_1}{\partial w}$ is continuous with respect to w in X^s. This implies that H_1 is C^1 with respect to w, and concludes the proof. □

Let us study now some properties of the derivative of H. In particular, consider first the operator

$$Tv := \frac{\partial H_1}{\partial w}(\mu,\xi,0,0,0)[v] = v - J(A'_0(z_{\mu,\xi})v). \tag{4.2.28}$$

This definition is well posed, as the next result points out.

Lemma 4.2.4. *T is a bounded operator from $\dot H^s(\mathbb{R}^N)$ to $\dot H^s(\mathbb{R}^N)$.*

Proof. Let $\psi := A_0'(z_{\mu,\xi})v = pz_{\mu,\xi}^{p-1}v$. From (4.1.13), we know that

$$[J(A_0'(z_{\mu,\xi})v)]_{\dot H^s(\mathbb{R}^N)} = [J\psi]_{\dot H^s(\mathbb{R}^N)} \leqslant C\,\|\psi\|_{L^\beta(\mathbb{R}^N)} = Cp\,\|z_{\mu,\xi}^{p-1}v\|_{L^\beta(\mathbb{R}^N)},$$

with $\beta := 2N/(N+2s)$. On the other hand, using the Hölder inequality with exponents $2_s^*/\beta$ and $(N+2s)/4s$ we can bound the quantity $\|z_{\mu,\xi}^{p-1}v\|_{L^\beta(\mathbb{R}^N)}$ with $C\,\|v\|_{L^{2_s^*}(\mathbb{R}^N)}$ and thus by $C\,[v]_{\dot H^s(\mathbb{R}^N)}$, thanks to the Sobolev inequality. This gives that

$$[J(A_0'(z_{\mu,\xi})v)]_{\dot H^s(\mathbb{R}^N)} \leqslant C\,[v]_{\dot H^s(\mathbb{R}^N)},$$

which implies the desired result. □

It is important to remark that T is also a linear operator over X^s. Of course, since X^s is a subset of $\dot H^s(\mathbb{R}^N)$, the restriction operator, that we still denote by T, maps X^s continuously to $\dot H^s(\mathbb{R}^N)$. What is relevant for us is that it also maps X^s continuously to X^s, as next result explicitly states.

Lemma 4.2.5. *T is a bounded operator from X^s to X^s.*

Proof. Same as the one of Lemma 4.2.4, using (4.1.15) in addition to (4.1.13). □

As a matter of fact, T enjoys further compactness properties, as observed in the next result.

Proposition 4.2.6. *T is a Fredholm operator over $\dot H^s(\mathbb{R}^N)$. More explicitly, if we set $Kv := -J(A_0'(z_{\mu,\xi})v)$, we have that $T = Id_{\dot H^s(\mathbb{R}^N)} + K$, and $K : \dot H^s(\mathbb{R}^N) \to \dot H^s(\mathbb{R}^N)$ is a compact operator over $\dot H^s(\mathbb{R}^N)$.*

Proof. We already know from Lemma 4.2.4 that K is a bounded operator over $\dot H^s(\mathbb{R}^N)$. Now, let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence such that

$$[v_k]_{\dot H^s(\mathbb{R}^N)} \leqslant 1. \tag{4.2.29}$$

To prove compactness, we need to see that

$$\{Kv_k\}_{k\in\mathbb{N}} \text{ contains a Cauchy subsequence in } \dot H^s(\mathbb{R}^N). \tag{4.2.30}$$

For this, we fix $\varepsilon > 0$ and we exploit (4.1.13) of Theorem 4.1.4 to obtain that

$$\begin{aligned}
&[Kv_l - Kv_m]_{\dot H^s(\mathbb{R}^N)}\\
&\quad = [J(A_0'(z_{\mu,\xi})(v_l - v_m))]_{\dot H^s(\mathbb{R}^N)}\\
&\quad \leqslant C\|A_0'(z_{\mu,\xi})(v_l - v_m)\|_{L^\beta(\mathbb{R}^N)}\\
&\quad = C(\|A_0'(z_{\mu,\xi})(v_l - v_m)\|_{L^\beta(B_R)} + \|A_0'(z_{\mu,\xi})(v_l - v_m)\|_{L^\beta(\mathbb{R}^N\setminus B_R)}),
\end{aligned} \tag{4.2.31}$$

where $\beta := \frac{2N}{N+2s}$, $R > 0$, and $B_R := \{x \in \mathbb{R}^N : |x| < R\}$.

Thus we notice that, for a fixed $R>0$, the quantity $\|v_k\|_{L^2(B_R)}$ is bounded by $\|v_k\|_{L^{2^*_s}(B_R)}$, by Hölder inequality, and the latter quantity is in turn bounded by $[v_k]_{\dot{H}^s(\mathbb{R}^N)}$, by Sobolev inequality. These observations and (4.2.29) imply that

$$\|v_k\|_{W^{s,2}(B_R)} \leqslant C_R,$$

for some $C_R>0$ that does not depend on k. Moreover, the space $W^{s,2}(B_R)$ is compactly embedded in $L^\beta(B_R)$ (see [82, Corollary 7.2] and recall that $\beta\in(1,2^*_s)$). This implies that v_k contains a Cauchy subsequence in $L^\beta(B_R)$ and so, up to a subsequence, if l and m are sufficiently large (say l, $m \geqslant N(R,\varepsilon)$, for some large $N(R,\varepsilon)$) we have that

$$\|v_l - v_m\|_{L^\beta(B_R)} \leqslant \varepsilon.$$

Notice also that

$$A_0'(z_{\mu,\xi}) = p z_{\mu,\xi}^{\frac{4s}{N-2s}} \in L^\infty(\mathbb{R}^N),$$

therefore

$$\|A_0'(z_{\mu,\xi})(v_l - v_m)\|_{L^\beta(B_R)} \leqslant \|A_0'(z_{\mu,\xi})\|_{L^\infty(\mathbb{R}^N)}\|v_l - v_m\|_{L^\beta(B_R)} \leqslant C\varepsilon \tag{4.2.32}$$

as long as l, $m \geqslant N(R,\varepsilon)$.

On the other hand, applying Hölder and Sobolev inequalities, and recalling (4.2.29) once again,

$$\begin{aligned}
&\|A_0'(z_{\mu,\xi})(v_l - v_m)\|_{L^\beta(\mathbb{R}^N\setminus B_R)}\\
&\qquad\leqslant \left(\int_{\mathbb{R}^N\setminus B_R} (v_l - v_m)^{2^*_s}\,dx\right)^{1/2^*_s}\left(\int_{\mathbb{R}^N\setminus B_R} (p z_{\mu,\xi}^{\frac{4s}{N-2s}})^{\frac{N}{2s}}\,dx\right)^{2s/n}\\
&\qquad\leqslant C\|v_l - v_m\|_{L^{2^*_s}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N\setminus B_{\frac{R-|\xi|}{\mu}}} \frac{1}{|y|^{2N}}\,dy\right)^{2s/n}\\
&\qquad\leqslant C[v_l - v_m]_{\dot{H}^s(\mathbb{R}^N)} R^{-N}\\
&\qquad\leqslant CR^{-N},
\end{aligned}$$

with $C>0$ possibly different from line to line, but independent of R, l and m. Thus, we insert this and (4.2.32) into (4.2.31) and we deduce that

$$[Kv_l - Kv_m]_{\dot{H}^s(\mathbb{R}^N)} \leqslant C(\varepsilon + R^{-N}),$$

provided that l, $m \geqslant N(R,\varepsilon)$, possibly up to a subsequence. In particular, we can choose R depending on ε, for instance $R := \varepsilon^{-1/N}$, and define $M_\varepsilon := M(\varepsilon^{-1/N},\varepsilon)$. So we obtain that, for l, $m \geqslant M_\varepsilon$, the quantity $[Kv_l - Kv_m]_{\dot{H}^s(\mathbb{R}^N)}$ is bounded by a constant times ε. This establishes (4.2.30). $\square$

Finally, for any $(v,\beta)\in \dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$ we define the linear operator

$$\mathcal{T}(v,\beta) := \left(Tv - \sum_{i=1}^{N+1} \beta_i q_i, \langle v, q_1\rangle, \dots, \langle v, q_{N+1}\rangle \right), \tag{4.2.33}$$

with T defined in (4.2.28). The interest of such operator for us is that

$$\frac{\partial H}{\partial(w,\alpha)}(\mu,\xi,0,0,0)[v,\beta] = \mathcal{T}(v,\beta). \tag{4.2.34}$$

We have:

Proposition 4.2.7. *$\mathcal{T}$ is a bounded operator from $\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$ to $\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$, and from $X^s\times\mathbb{R}^{N+1}$ to $X^s\times\mathbb{R}^{N+1}$.*

Furthermore, $\mathcal{T}$ is a Fredholm operator over $\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$. More explicitly, it can be written as the identity plus a compact operator over $\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$.

Proof. Let

$$S(v,\beta) := \left(-\sum_{i=1}^{N+1} \beta_i q_i, \langle v, q_1\rangle, \dots, \langle v, q_{N+1}\rangle \right).$$

Let also $\|\cdot\|$ be either $\|\cdot\|_{\dot H^s(\mathbb{R}^N)}$ or $\|\cdot\|_{X^s}$. We have that

$$\begin{aligned}\|S(v,\beta)\| &\leqslant \sum_{i=1}^{N+1} |\beta_i|\,\|q_i\| + \sum_{i=1}^{N+1} \|v\|_{\dot H^s(\mathbb{R}^N)} \|q_i\|_{\dot H^s(\mathbb{R}^N)} \\ &\leqslant C\Big(|\beta| + \|v\|_{\dot H^s(\mathbb{R}^N)}\Big) \\ &\leqslant C\Big(|\beta| + \|v\|\Big).\end{aligned}$$

This shows that S is a bounded operator from $\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$ to $\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$, and from $X^s\times\mathbb{R}^{N+1}$ to $X^s\times\mathbb{R}^{N+1}$. Then, noticing that $\mathcal{T} = (T,0) + S$ and recalling Lemmata 4.2.4 and 4.2.5, we obtain that also $\mathcal{T}$ is a bounded operator from $\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$ to $\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$, and from $X^s\times\mathbb{R}^{N+1}$ to $X^s\times\mathbb{R}^{N+1}$.

Now we show that it is Fredholm over $\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$. For this, we set

$$\mathcal{K}(v,\beta) := \left(Kv - \sum_{i=1}^{N+1} \beta_i q_i, \langle v, q_1\rangle - \beta_1, \dots, \langle v, q_{N+1}\rangle - \beta_{n+1} \right),$$

where K is the operator in Proposition 4.2.6. Notice that $\mathcal{T} = Id_{\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}} + \mathcal{K}$, so our goal is to show that $\mathcal{K}$ is compact over $\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$. For this, we take a sequence $(v_k,\beta_k)\in \dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$ with $\|v_k\|_{\dot H^s(\mathbb{R}^N)} + \|\beta_k\|_{\mathbb{R}^{N+1}} \leqslant 1$ and we want to find a Cauchy subsequence of $\mathcal{T}(v_k,\beta_k)$ in $\dot H^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$.

To this goal, we use Proposition 4.2.6 to obtain a subsequence (still denoted by v_k) such that Kv_k is Cauchy in $\dot H^s(\mathbb{R}^N)$. Also, again up to subsequences, v_k is weakly convergent in $\dot{H}^s(\mathbb{R}^N)$, therefore $\langle v_k, q_1\rangle$ is Cauchy (and the same holds for $\langle v_k, q_2\rangle, \dots, \langle v_k, q_{N+1}\rangle$). Finally, since $\mathbb{R}^{N+1}$ is finite dimensional, up to subsequence we can assume that also β_k is Cauchy. Thanks to these considerations, and writing $\beta_k = (\beta_{k,1}, \dots, \beta_{k,n+1}) \in \mathbb{R}^{N+1}$, we have that

$$\begin{aligned} &\big\|\mathcal{K}(v_k, \beta_k) - \mathcal{K}(v_m, \beta_m)\big\|_{\dot{H}^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}} \\ &\qquad \leqslant \|Kv_k - Kv_m\|_{\dot{H}^s(\mathbb{R}^N)} + \sum_{i=1}^{N+1} |\beta_{k,i} - \beta_{m,i}|\, \|q_i\|_{\dot{H}^s(\mathbb{R}^N)} + \sum_{i=1}^{N+1} |\langle v_k - v_m,\, q_i\rangle| \\ &\qquad \leqslant C\left(\|Kv_k - Kv_m\|_{\dot{H}^s(\mathbb{R}^N)} + \|\beta_k - \beta_m\|_{\mathbb{R}^{N+1}} + \sum_{i=1}^{N+1} |\langle v_k - v_m,\, q_i\rangle|\right) \\ &\qquad \leqslant \varepsilon, \end{aligned}$$

provided that k and m are large enough. This shows that (v_k, β_k) is Cauchy, as desired. □

Invertibility issues.

Now we discuss the invertibility of the operator $\mathcal{T}$ that was introduced in (4.2.33). Notice that there is a subtle point here. Indeed, the operator $\mathcal{T}$ can be seen as acting over $\dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1}$ or over $X^s \times \mathbb{R}^{N+1}$ (see Proposition 4.2.7). On the one hand, the invertibility over $\dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1}$ should be expected to be easier, since the operator is Fredholm there (see the last claim in Proposition 4.2.7). On the other hand, since we want to obtain strong pointwise estimates to keep control of the possible singularities of our functional, it is crucial for us to invert the operator in a space that controls the functions uniformly, namely $X^s \times \mathbb{R}^{N+1}$. So our strategy will be the following: first we invert the operator in $\dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1}$ (this will be accomplished using the Fredholm property in Proposition 4.2.7, the regularity theory in Theorem 4.1.6 and a nondegeneracy result in [79]). Then we will deduce from this information and a further regularity theory that $\mathcal{T}$ is actually invertible also in $X^s \times \mathbb{R}^{N+1}$.

The details of the argument go as follows. First, we recall the standard definition of invertibility:

Definition 4.2.8. Let X, Y Banach spaces, and let $S : X \to Y$ be a linear bounded operator. We say that S is invertible (and we write $S \in Inv(X, Y)$) if there exists a linear bounded operator $\tilde{S} : Y \to X$ such that

$$S\tilde{S} = Id_Y, \quad \tilde{S}S = Id_X.$$

Then, we show that $\mathcal{T}$ is invertible in $\dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1}$:

Proposition 4.2.9. $\mathcal{T} \in Inv(\dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1}, \dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1})$.

Proof. By Proposition 4.2.7 and the theory of Fredholm operators (see e.g. [46, pages 168-169], for a very brief summary, and [126, Chapter IV, Section 5], or [152], for a detailed analysis), it is enough to show that $\mathcal{T}$ is injective over $\dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1}$. For this, let us take $(v,\beta) \in \dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1}$ such that $\mathcal{T}(v,\beta) = 0$, that is, by (4.2.33),

$$\begin{aligned} Tv &= \sum_{i=1}^{N+1} \beta_i q_i, \\ \langle v, q_1\rangle &= \cdots = \langle v, q_{N+1}\rangle = 0. \end{aligned} \tag{4.2.35}$$

Fixed $j \in \{1,\ldots,N+1\}$, using (4.2.28), (4.1.14) and (4.2.5), we observe that

$$\begin{aligned} \langle Tv, q_j\rangle &= \langle v - pJ(z_{\mu,\xi}^{p-1} v),\, q_j\rangle = \langle v,\, q_j\rangle - p\int_{\mathbb{R}^N} (-\Delta)^s J(z_{\mu,\xi}^{p-1} v)\, q_j \\ &= \langle v,\, q_j\rangle - p\int_{\mathbb{R}^N} z_{\mu,\xi}^{p-1} v\, q_j = \langle v,\, q_j\rangle - \int_{\mathbb{R}^N} v\,(-\Delta)^s q_j \\ &= \langle v,\, q_j\rangle - \langle v,\, q_j\rangle = 0. \end{aligned} \tag{4.2.36}$$

This, (4.2.35) and Lemma 4.2.1 give that

$$0 = \langle Tv, q_j\rangle = \sum_{i=1}^{N+1} \beta_i \langle q_i, q_j\rangle = \lambda_j \beta_j,$$

and so

$$\beta_j = 0 \text{ for every } j \in \{1,\ldots,N+1\}. \tag{4.2.37}$$

Therefore, $v \in \dot{H}^s(\mathbb{R}^N)$ is an energy solution of $Tv = 0$, that is, by (4.2.28) and (4.1.14), the equation $(-\Delta)^s v = p z_{\mu,\xi}^{p-1} v$. Accordingly, by Theorem 4.1.6, we obtain that v belongs to $L^\infty(\mathbb{R}^N)$.

Thanks to this, we can apply the nondegeneracy result in [79], that gives that v must be a linear combination of $q_1,\ldots,q_{N+1}$. So we write

$$v = \sum_{i=1}^{N+1} c_i q_i \tag{4.2.38}$$

for some $c_i \in \mathbb{R}$, we recall (4.2.35) and once again Lemma 4.2.1, and we compute

$$0 = \langle v, q_j\rangle = \sum_{i=1}^{N+1} c_i \langle q_i, q_j\rangle = c_j \lambda_j,$$

that gives $c_j = 0$ for every $j \in \{1,\ldots,N+1\}$. By plugging this information into (4.2.38), we conclude that $v = 0$. This and (4.2.37) give that $(v,\beta) = 0$ and so $\mathcal{T}$ is injective on $\dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1}$. □

Next, we aim to prove that $\mathcal{T} \in Inv(X^s \times \mathbb{R}^{N+1},\, X^s \times \mathbb{R}^{N+1})$. For this scope, we need an improved regularity theory result, which goes as follows:

Lemma 4.2.10. *Let $C_0 > 0$. For any $u \in X^s$, $(\alpha, \beta) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ and any $\psi \in \dot{H}^s(\mathbb{R}^N)$ which is an energy solution of*

$$(-\Delta)^s \psi = p \sum_{i=1}^{N+1} \alpha_i z_{\mu,\xi}^{p-1} q_i + p z_{\mu,\xi}^{p-1} \psi + p z_{\mu,\xi}^{p-1} u \tag{4.2.39}$$

with

$$[\psi]_{\dot{H}^s(\mathbb{R}^N)} \leqslant C_0 \Big(\|u\|_{X^s} + \|\beta\|_{\mathbb{R}^{N+1}} \Big), \tag{4.2.40}$$

we have that $\psi \in L^\infty(\mathbb{R}^N)$ and

$$\|\psi\|_{L^\infty(\mathbb{R}^N)} \leqslant C \Big(\|u\|_{X^s} + \|\alpha\|_{\mathbb{R}^{N+1}} + \|\beta\|_{\mathbb{R}^{N+1}} \Big) \tag{4.2.41}$$

for some $C > 0$.

Proof. The core of the proof is that the equation is linear in the triplet (ψ, u, α), so we get the desired result by a careful scaling argument. The rigorous argument goes as follows. First, we use Theorem 4.1.6 to get that $\psi \in L^\infty(\mathbb{R}^N)$, so we focus on the proof of (4.2.41). Suppose, by contradiction, that (4.2.41) is false. Then, for any k there exists a quadruplet $(\psi_k, u_k, \alpha_k, \beta_k) \in \dot{H}^s(\mathbb{R}^N) \times X^s \times \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ such that

$$(-\Delta)^s \psi_k = p \sum_{i=1}^{N+1} \alpha_{k,i} z_{\mu,\xi}^{p-1} q_i + p z_{\mu,\xi}^{p-1} \psi_k + p z_{\mu,\xi}^{p-1} u_k, \tag{4.2.42}$$

$$\|\psi_k\|_{L^\infty(\mathbb{R}^N)} > k \left(\|u_k\|_{X^s} + \|\alpha_k\|_{\mathbb{R}^{N+1}} + \|\beta_k\|_{\mathbb{R}^{N+1}} \right) \tag{4.2.43}$$

and

$$[\psi_k]_{\dot{H}^s(\mathbb{R}^N)} \leqslant C_0 \Big(\|u_k\|_{X^s} + \|\beta_k\|_{\mathbb{R}^{N+1}} \Big). \tag{4.2.44}$$

We remark that $\psi_k \in L^\infty(\mathbb{R}^N)$ and $\|\psi_k\|_{L^\infty(\mathbb{R}^N)} > 0$, due to (4.2.43). Thus, we can define

$$\tilde{\psi}_k := \frac{\psi_k}{\|\psi_k\|_{L^\infty(\mathbb{R}^N)}}, \qquad \tilde{u}_k := \frac{u_k}{\|\psi_k\|_{L^\infty(\mathbb{R}^N)}},$$
$$\tilde{\alpha}_k := \frac{\alpha_k}{\|\psi_k\|_{L^\infty(\mathbb{R}^N)}} \qquad \text{and} \qquad \tilde{\beta}_k := \frac{\beta_k}{\|\psi_k\|_{L^\infty(\mathbb{R}^N)}}.$$

Notice that

$$\begin{aligned} & \|\tilde{\psi}_k\|_{L^\infty(\mathbb{R}^N)} = 1, \\ \text{and} \quad & \|\tilde{u}_k\|_{X^s} + \|\tilde{\alpha}_k\|_{\mathbb{R}^{N+1}} + \|\tilde{\beta}_k\|_{\mathbb{R}^{N+1}} = \frac{\|u_k\|_{X^s} + \|\alpha_k\|_{\mathbb{R}^{N+1}} + \|\beta_k\|_{\mathbb{R}^{N+1}}}{\|\psi_k\|_{L^\infty(\mathbb{R}^N)}} \leqslant \frac{1}{k}, \end{aligned} \tag{4.2.45}$$

thanks to (4.2.43).

Also, by linearity, equation (4.2.42) becomes

$$(-\Delta)^s\tilde{\psi}_k = p\sum_{i=1}^{N+1}\tilde{\alpha}_{k,i}z_{\mu,\xi}^{p-1}q_i + pz_{\mu,\xi}^{p-1}\tilde{\psi}_k + pz_{\mu,\xi}^{p-1}\tilde{u}_k.$$

The right hand side of this equation is bounded uniformly in $L^\infty(\mathbb{R}^N)$, thanks to (4.2.45) and the fact that $z_{\mu,\xi}, q_i \in L^\infty(\mathbb{R}^N)$, $i \in \{1, \dots, N+1\}$.

Thus, by Proposition 5 in [158], we know that for every $x \in \mathbb{R}^N$, there exists a constant $C > 0$ and $a \in (0,1)$ such that

$$\|\tilde{\psi}_k\|_{C^a(B_{1/4}(x))} \leqslant C.$$

We remark that C and a here are independent of k and x, therefore

$$\|\tilde{\psi}_k\|_{C^a(\mathbb{R}^N)} \leqslant C. \tag{4.2.46}$$

From (4.2.45), we know that there exists a point $x_k \in \mathbb{R}^N$ such that $\tilde{\psi}_k(x_k) \geqslant 1/2$. By (4.2.46), there exists $\rho > 0$, which is independent of k, such that $\tilde{\psi}_k \geqslant 1/4$ in $B_\rho(x_k)$. As a consequence,

$$\|\tilde{\psi}_k\|_{L^{2_s^*}(\mathbb{R}^N)} \geqslant \left(\int_{B_\rho(x_k)} \left(\frac{1}{4}\right)^{2_s^*} dx\right)^{1/2_s^*} \geqslant c_0,$$

with $c_0 > 0$ independent of k. Thus, by Sobolev inequality,

$$[\tilde{\psi}_k]_{\dot{H}^s(\mathbb{R}^N)} \geqslant c_0, \tag{4.2.47}$$

up to renaming c_0. On the other hand, by (4.2.44) and (4.2.43), we have that

$$[\tilde{\psi}_k]_{\dot{H}^s(\mathbb{R}^N)} = \frac{[\psi_k]_{\dot{H}^s(\mathbb{R}^N)}}{\|\psi_k\|_{L^\infty(\mathbb{R}^N)}} \leqslant \frac{C_0\Big(\|u_k\|_{X^s} + \|\beta_k\|_{\mathbb{R}^{N+1}}\Big)}{\|\psi_k\|_{L^\infty(\mathbb{R}^N)}} \leqslant \frac{C_0}{k}.$$

This is in contradiction with (4.2.47) when k is large, and therefore the desired result is established. □

Finally, we show that $\mathcal{T}$ is invertible in $X^s(\mathbb{R}^N) \times \mathbb{R}^{N+1}$:

Proposition 4.2.11. *$\mathcal{T} \in Inv(X^s \times \mathbb{R}^{N+1}, X^s \times \mathbb{R}^{N+1})$.*

Proof. By Proposition 4.2.9, we know that $\mathcal{T} \in Inv(\dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1}, \dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1})$. Therefore, there exists an operator

$$\tilde{\mathcal{T}} : \dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1} \to \dot{H}^s(\mathbb{R}^N) \times \mathbb{R}^{N+1},$$

that is linear and bounded and such that $\mathcal{T}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}\mathcal{T} = Id_{\dot{H}^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}}$. The boundedness of $\tilde{\mathcal{T}}$ as an operator acting over $\dot{H}^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$ can be explicitly written as

$$\|\tilde{\mathcal{T}}(u,\beta)\|_{\dot{H}^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}} \leqslant C\,\|(u,\beta)\|_{\dot{H}^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}}. \tag{4.2.48}$$

Now, since X^s is a subset of $\dot{H}^s(\mathbb{R}^N)$, we can consider the restriction operator of $\tilde{\mathcal{T}}$ acting on $X^s\times\mathbb{R}^{N+1}$ (this restriction operator will be denoted by $\tilde{\mathcal{T}}$ as well). We observe that, for any $u\in X^s$, we have that $u\in\dot{H}^s(\mathbb{R}^N)$, therefore, for any $\beta\in\mathbb{R}^{N+1}$,

$$\mathcal{T}\tilde{\mathcal{T}}(u,\beta) = Id_{\dot{H}^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}}(u,\beta) = (u,\beta).$$

Furthermore, if $u\in X^s$ and $\beta\in\mathbb{R}^{N+1}$, then $\mathcal{T}(u,\beta)\in X^s\times\mathbb{R}^{N+1}$, due to Proposition 4.2.7. Hence the restriction of $\tilde{\mathcal{T}}$ over $X^s\times\mathbb{R}^{N+1}$ may act on $\mathcal{T}(u,\beta)$, for any $(u,\beta)\in X^s\times\mathbb{R}^{N+1}$, and we obtain that

$$\tilde{\mathcal{T}}\mathcal{T}(u,\beta) = Id_{\dot{H}^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}}(u,\beta) = (u,\beta).$$

It remains to prove that

$$\|\tilde{\mathcal{T}}(u,\beta)\|_{X^s\times\mathbb{R}^{N+1}} \leqslant C\,\Big(\|u\|_{X^s} + \|\beta\|_{\mathbb{R}^{N+1}}\Big). \tag{4.2.49}$$

To prove it, we first use (4.2.48) to bound $\|\tilde{\mathcal{T}}(u,\beta)\|_{\dot{H}^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}}$ with $[u]_{\dot{H}^s(\mathbb{R}^N)}+\|\beta\|_{\mathbb{R}^{N+1}}$, and then we observe that the latter quantity is in turn bounded by $\|u\|_{X^s}+\|\beta\|_{\mathbb{R}^{N+1}}$. Thus, in order to show that $\mathcal{T}$ is bounded as an operator over $X^s\times\mathbb{R}^{N+1}$, we only have to bound $\|\tilde{\mathcal{T}}(u,\beta)\|_{L^\infty(\mathbb{R}^N)\times\mathbb{R}^{N+1}}$.

That is to say that the desired result is proved if we show that, for any $u\in X^s$ and any $\beta\in\mathbb{R}^{N+1}$ we have that

$$\|\tilde{\mathcal{T}}(u,\beta)\|_{L^\infty(\mathbb{R}^N)\times\mathbb{R}^{N+1}} \leqslant C\,\Big(\|u\|_{X^s} + \|\beta\|_{\mathbb{R}^{N+1}}\Big). \tag{4.2.50}$$

To prove this, we fix $u\in X^s$ and $\beta\in\mathbb{R}^{N+1}$ and we set $(v,\alpha):=\tilde{\mathcal{T}}(u,\beta)\in\dot{H}^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}$. Thus, by (4.2.33),

$$X^s\times\mathbb{R}^{N+1}\ni(u,\beta)=\mathcal{T}(v,\alpha)=\left(Tv-\sum_{i=1}^{N+1}\alpha_i q_i,\langle v,q_1\rangle,\dots,\langle v,q_{N+1}\rangle\right). \tag{4.2.51}$$

Taking the first coordinate and using (4.2.36), we obtain that, for any $j\in\{1,\dots,N+1\}$,

$$\langle u,q_j\rangle = \langle Tv-\sum_{i=1}^{N+1}\alpha_i q_i,\, q_j\rangle = -\sum_{i=1}^{N+1}\alpha_i\langle q_i,\, q_j\rangle.$$

Thus, by Lemma 4.2.1, we have that $\langle u,q_j\rangle = -\alpha_j\lambda_j$ and therefore

$$|\alpha_j|\leqslant C\,[u]_{\dot{H}^s(\mathbb{R}^N)}.$$

Accordingly

$$\|\alpha\|_{\mathbb{R}^{N+1}} \leqslant C\,\|u\|_{X^s}. \tag{4.2.52}$$

Now we set $\psi := v - u$. Notice that $\psi \in \dot{H}^s(\mathbb{R}^N)$, since so are u and v. Moreover, taking the first coordinate in (4.2.51) and using (4.2.28) and (4.1.14), we see that ψ is an energy solution of

$$\begin{aligned}(-\Delta)^s\psi &= (-\Delta)^s v - (-\Delta)^s u\\ &= (-\Delta)^s v - (-\Delta)^s Tv + \sum_{i=1}^{N+1}\alpha_i(-\Delta)^s q_i\\ &= (-\Delta)^s J(A_0'(z_{\mu,\xi})v) + \sum_{i=1}^{N+1}\alpha_i(-\Delta)^s q_i\\ &= pz_{\mu,\xi}^{p-1}v + p\sum_{i=1}^{N+1}\alpha_i z_{\mu,\xi}^{p-1}q_i\\ &= pz_{\mu,\xi}^{p-1}\psi + pz_{\mu,\xi}^{p-1}u + p\sum_{i=1}^{N+1}\alpha_i z_{\mu,\xi}^{p-1}q_i.\end{aligned}$$

The reader may check that this agrees with (4.2.39). Furthermore, by (4.2.48),

$$\begin{aligned}[v]_{\dot{H}^s(\mathbb{R}^N)} &\leqslant \|(v,\alpha)\|_{\dot{H}^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}}\\ &= \|\tilde{\mathcal{T}}(u,\beta)\|_{\dot{H}^s(\mathbb{R}^N)\times\mathbb{R}^{N+1}}\\ &\leqslant C\Big([u]_{\dot{H}^s(\mathbb{R}^N)} + \|\beta\|_{\mathbb{R}^{N+1}}\Big).\end{aligned}$$

Consequently,

$$[\psi]_{\dot{H}^s(\mathbb{R}^N)} \leqslant [u]_{\dot{H}^s(\mathbb{R}^N)} + [v]_{\dot{H}^s(\mathbb{R}^N)} \leqslant C\Big([u]_{\dot{H}^s(\mathbb{R}^N)} + \|\beta\|_{\mathbb{R}^{N+1}}\Big),$$

up to renaming constants. The reader may check that this implies (4.2.40). Accordingly the assumptions of Lemma 4.2.10 are satisfied, and we deduce from it that

$$\|\psi\|_{L^\infty(\mathbb{R}^N)} \leqslant C\Big(\|u\|_{X^s} + \|\alpha\|_{\mathbb{R}^{N+1}} + \|\beta\|_{\mathbb{R}^{N+1}}\Big).$$

Therefore, using (4.2.52), we obtain that

$$\begin{aligned}\|v\|_{L^\infty(\mathbb{R}^N)} &\leqslant \|u\|_{L^\infty(\mathbb{R}^N)} + \|\psi\|_{L^\infty(\mathbb{R}^N)}\\ &\leqslant C\left(\|u\|_{X^s} + \|\alpha\|_{\mathbb{R}^{N+1}} + \|\beta\|_{\mathbb{R}^{N+1}}\right)\\ &\leqslant C\left(\|u\|_{X^s} + \|\beta\|_{\mathbb{R}^{N+1}}\right),\end{aligned}$$

up to renaming constants. Using this and once again (4.2.52), we obtain that

$$\begin{aligned}&\|\tilde{\mathcal{T}}(u,\beta)\|_{L^\infty(\mathbb{R}^N)\times\mathbb{R}^{N+1}} = \|(v,\alpha)\|_{L^\infty(\mathbb{R}^N)\times\mathbb{R}^{N+1}}\\ &\qquad = \|v\|_{L^\infty(\mathbb{R}^N)} + \|\alpha\|_{\mathbb{R}^{N+1}} \leqslant C\left(\|u\|_{X^s} + \|\beta\|_{\mathbb{R}^{N+1}}\right).\end{aligned}$$

This establishes (4.2.50) and in turn (4.2.49), and so it completes the proof of the desired result. □

Proof of Lemma 4.2.2.

Once we have studied in detail the operator H, we can prove Lemma 4.2.2. As we pointed out at the beginning of this subsection, the idea is to do it by means of the Implicit Function Theorem. For the sake of completeness, we write here the precise statement of this theorem that we will use (see [24, Theorem 2.3, page 38]).

Theorem 4.2.12. *(Implicit Function Theorem)*
Let X, Y, Z be Banach spaces, and let Λ and U be open sets of X and Y respectively. Let $H \in C^1(\Lambda \times U, Z)$ and suppose that

$$H(\lambda^*, u^*) = 0 \quad \textit{and} \quad \frac{\partial H}{\partial u}(\lambda^*, u^*) \in Inv(Y, Z).$$

Then there exist neighborhoods Θ of λ^* in X and U^* of u^* in Y, and a map $g \in C^1(\Theta, Y)$ such that

(a) $H(\lambda, g(\lambda)) = 0$, for all $\lambda \in \Theta$,

(b) $H(\lambda, u) = 0$, with $(\lambda, u) \in \Theta \times U^*$, implies $u = g(\lambda)$,

(c) $g'(\lambda) = -\left(\frac{\partial H}{\partial u}(p)\right)^{-1} \circ \frac{\partial H}{\partial \lambda}(p)$, where $p = (\lambda, g(\lambda))$ and $\lambda \in \Theta$.

Now we conclude the proof of Lemma 4.2.2.

Proof of Lemma 4.2.2. Consider H defined in (4.2.15). First we observe that H is C^1 with respect to μ and ξ. Indeed, $z_{\mu,\xi}$ is C^1 with respect to μ and ξ. Moreover, J is linear and $A_\varepsilon(z_{\mu,\xi} + w)$ is C^1 with respect to $z_{\mu,\xi}$ since $z_{\mu,\xi} + w$ is bounded from zero on the support of h (recall (4.2.3)), therefore H_1 is C^1 with respect to $z_{\mu,\xi}$.

Also, H is C^1 with respect to ε and α, since it depends linearly on these variables (recall that J is linear and A_ε is linear with respect to ε). Finally, H is C^1 with respect to w thanks to Lemma 4.2.3.

Now we use the Implicit Function Theorem. Indeed, we notice that

$$H_1(\mu, \xi, 0, 0, 0) = z_{\mu,\xi} - J(A_0(z_{\mu,\xi})) = z_{\mu,\xi} - J(z^p_{\mu,\xi}) = 0, \tag{4.2.53}$$

since $z_{\mu,\xi}$ is a solution to (4.1.1) (recall also (4.1.14)). Moreover,

$$H_2(\mu, \xi, 0, 0, 0) = 0. \tag{4.2.54}$$

In order to follow the notation of Theorem 4.2.12, we set

$$X := \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}, \quad Y := X^s \times \mathbb{R}^{N+1}, \quad Z := X^s \times \mathbb{R}^{N+1},$$

$$\Lambda := (\mu_1, \mu_2) \times B_R \times \mathbb{R}, \quad U := V \times \mathbb{R}^{N+1},$$

and

$$\lambda^* := (\mu, \xi, 0), \quad u^* := (0, 0), \quad u := (w, \alpha).$$

Thus, we have proved that

(i) $H \in C^1(\Lambda \times U, Z)$, by the linear dependance of the variables and Lemma 4.2.3;

(ii) $H(\lambda^*, u^*) = 0$, by (4.2.53) and (4.2.54);

(iii) $\dfrac{\partial H}{\partial u}(\lambda^*, u^*) \in Inv(Y, Z)$, by (4.2.33), (4.2.34) and Proposition 4.2.11.

Notice here that, since V was defined as

$$V := \{w \in X^s \text{ s.t } \|w\|_{X^s} < a/2\},$$

it is an open subset of X^s. Therefore, all the hypotheses of the Implicit Function Theorem are satisfied, and we conclude the existence of $w \in X^s$ solution to (4.2.16), that is, there exists $w \in X^s \cap \left(T_{z_{\mu,\xi}} Z_0\right)^\perp$ that solves the auxiliary equation in (4.2.10). Furthermore, since H is of class C^1 with respect to ε, μ and ξ in X^s, we deduce that so is w.

Now we focus on the proof of (4.2.11). We observe that

$$\left\| \frac{\partial(w, \alpha)}{\partial \varepsilon} \right\|_{X^s \times \mathbb{R}^{N+1}} \leqslant C. \tag{4.2.55}$$

Indeed, we write

$$H\big(\mu,\, \xi,\, \varepsilon,\, w(\varepsilon, z_{\mu,\xi}),\, \alpha(\varepsilon, z_{\mu,\xi})\big) = 0, \tag{4.2.56}$$

we differentiate with respect to ε and we set $\varepsilon := 0$. Since

$$w(0, z_{\mu,\xi}) = 0 \text{ and } \alpha(0, z_{\mu,\xi}) = 0, \tag{4.2.57}$$

we obtain that

$$\frac{\partial H}{\partial \varepsilon}(\mu, \xi, 0, 0, 0) + \frac{\partial H}{\partial(w, \alpha)}(\mu, \xi, 0, 0, 0) \frac{\partial(w, \alpha)}{\partial \varepsilon}(0, z_{\mu,\xi}) = 0.$$

Therefore, using the invertibility assumption, we get that

$$\frac{\partial(w, \alpha)}{\partial \varepsilon}(0, z_{\mu,\xi}) = -\left(\frac{\partial H}{\partial(w, \alpha)}(\mu, \xi, 0, 0, 0)\right)^{-1} \frac{\partial H}{\partial \varepsilon}(\mu, \xi, 0, 0, 0),$$

and so, since H is C^1 with respect to X^s,

$$\left\| \frac{\partial(w, \alpha)}{\partial \varepsilon}(0, z_{\mu,\xi}) \right\|_{X^s \times \mathbb{R}^{N+1}} \leqslant C.$$

Then, since (w, α) is C^1 in ε, in virtue of the Implicit Function Theorem, we obtain (4.2.55).

From (4.2.55) and (4.2.57) we obtain that

$$\|(w,\alpha)\|_{X^s\times\mathbb{R}^{N+1}} \leqslant C\varepsilon,$$

and this implies the first estimate in (4.2.11).

Now we prove the second and third estimates in (4.2.11). In this case, we will see that the roles of μ and ξ are basically the same: for this, we write $\varpi \in \mathbb{R}$ for any of the variables $(\mu,\xi) \in \mathbb{R}^{N+1}$ and we use the linearized equation to see that

$$(-\Delta)^s \frac{\partial z_{\mu,\xi}}{\partial \varpi} = p z_{\mu,\xi}^{p-1} \frac{\partial z_{\mu,\xi}}{\partial \varpi}.$$

This information can be written as

$$\frac{\partial H}{\partial \varpi}(\mu,\xi,0,0,0) = 0.$$

Now we take derivatives of (4.2.56) with respect to ϖ and we set $\varepsilon := 0$. Recalling (4.2.57) we obtain that

$$\begin{aligned} 0 &= \frac{\partial H}{\partial \varpi}(\mu,\xi,0,0,0) + \frac{\partial H}{\partial (w,\alpha)}(\mu,\xi,0,0,0)\frac{\partial (w,\alpha)}{\partial \varpi}(0,z_{\mu,\xi}) \\ &= \frac{\partial H}{\partial (w,\alpha)}(\mu,\xi,0,0,0)\frac{\partial (w,\alpha)}{\partial \varpi}(0,z_{\mu,\xi}). \end{aligned}$$

Hence, from the invertibility condition, we conclude that

$$\frac{\partial (w,\alpha)}{\partial \varpi}(0,z_{\mu,\xi}) = 0.$$

Since (w,α) are C^1 in ε, we obtain that

$$\lim_{\varepsilon\to 0} \left\| \frac{\partial (w,\alpha)}{\partial \varpi}(\varepsilon,z_{\mu,\xi}) \right\|_{X^s\times\mathbb{R}^{N+1}} = 0.$$

This gives the second and third claim in (4.2.11) and completes the proof of Lemma 4.2.2. □

4.2.3 Finite-dimensional reduction.

Up to this point, we have found a function w so that $z_{\mu,\xi} + w$ satisfies our problem in the energy sense, when we test with functions $\varphi \in (T_{z_{\mu,\xi}} Z_0)^\perp \cap X^s$. The following result states that actually the equation is satisfied for every test function in X^s, i.e. that $z_{\mu,\xi} + w$ is a solution to (4.0.1).

Indeed, consider the reduced functional $\Phi_\varepsilon : Z_0 \to \mathbb{R}$, defined by

$$\Phi_\varepsilon(z) := \mathcal{I}_\varepsilon(z+w),$$

where $w = w(\varepsilon, z)$ is provided by Lemma 4.2.2.

Proposition 4.2.13. *Suppose that Φ_ε has a critical point $z_{\mu^\varepsilon,\xi^\varepsilon} \in Z_0$ for ε small enough. Thus, $z_{\mu^\varepsilon,\xi^\varepsilon} + w$ is a critical point of $\mathcal{I}_\varepsilon$, where $w = w(\varepsilon, z_{\mu^\varepsilon,\xi^\varepsilon}) \in (T_{z_{\mu^\varepsilon,\xi^\varepsilon}} Z_0)^\perp$ is provided by Lemma 4.2.2.*

Proof. For simplicity, we will denote $\mu := \mu^\varepsilon$ and $\xi := \xi^\varepsilon$, and thus $z_{\mu,\xi} := z_{\mu^\varepsilon,\xi^\varepsilon}$. Since $z_{\mu,\xi}$ is a critical point of Φ_ε for ε small enough, we know that there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and every $\varphi \in (T_{z_{\mu,\xi}} Z_0) \cap X^s$ it holds

$$\frac{d}{dt}\Phi_\varepsilon(\psi(t))\bigg|_{t=0} = 0, \tag{4.2.58}$$

where $\psi : [0,1] \to Z_0$ is a curve in Z_0 such that $\psi(0) = z_{\mu,\xi}$ and $\psi'(0) = \varphi$. Recalling the definition of Φ_ε, we observe that

$$\begin{aligned}
\frac{d}{dt}\Phi_\varepsilon(\psi(t))\bigg|_{t=0} &= \frac{d}{dt}\mathcal{I}_\varepsilon(\psi(t) + w(\varepsilon, \psi(t)))\bigg|_{t=0} \\
&= \frac{d}{dt}\left[\mathcal{I}_\varepsilon(\psi(0) + w(\varepsilon,\psi(0))) + t\cdot \mathcal{I}_\varepsilon'(\psi(0) + w(\varepsilon,\psi(0)))\left(\psi'(0) + \frac{\partial w}{\partial z_{\mu,\xi}}\psi'(0)\right)\right]_{t=0} \\
&= \mathcal{I}_\varepsilon'(z_{\mu,\xi} + w(\varepsilon, z_{\mu,\xi}))\left(\varphi + \frac{\partial w}{\partial z_{\mu,\xi}}\varphi\right),
\end{aligned}$$

and hence (4.2.58) is equivalent to

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{\big((z_{\mu,\xi}+w)(x) - (z_{\mu,\xi}+w)(y)\big)\big((\varphi + \frac{\partial w}{\partial z_{\mu,\xi}}\varphi)(x) - (\varphi + \frac{\partial w}{\partial z_{\mu,\xi}}\varphi)(y)\big)}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad = \int_{\mathbb{R}^N}\Big(\varepsilon h(x)\big(z_{\mu,\xi}(x) + w(x)\big)^q + \big(z_{\mu,\xi}(x)+w(x)\big)^p\Big)\left(\varphi + \frac{\partial w}{\partial z_{\mu,\xi}}\varphi\right)(x)\,dx,
\end{aligned} \tag{4.2.59}$$

for any $\varphi \in \big(T_{z_{\mu,\xi}} Z_0\big) \cap X^s$.

Moreover, since w solves (4.2.16), $H_1(\mu,\xi,w,\varepsilon,\alpha) = 0$ is equivalent to affirm that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{\big((z_{\mu,\xi}+w)(x) - (z_{\mu,\xi}+w)(y)\big)\big(\phi(x)-\phi(y)\big)}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^N}\Big(\varepsilon h(x)\big(z_{\mu,\xi}(x)+w(x)\big)^q + \big(z_{\mu,\xi}(x)+w(x)\big)^p\Big)\phi(x)\,dx, \\
&\quad = \sum_{i=1}^{N+1}\alpha_i \iint_{\mathbb{R}^{2N}} \frac{\big(q_i(x)-q_i(y)\big)\big(\phi(x)-\phi(y)\big)}{|x-y|^{N+2s}}\,dx\,dy,
\end{aligned} \tag{4.2.60}$$

for any $\phi \in X^s$.

Consider now $q_j \in T_{z_{\mu,\xi}} Z_0$ defined in (4.2.4). Thus, taking $\varphi := q_j$ in (4.2.59) and applying (4.2.60) with $\phi := q_j + \frac{\partial w}{\partial z_{\mu,\xi}} q_j$ we obtain

$$
\begin{aligned}
0 &= \sum_{i=1}^{N+1} \alpha_i \iint_{\mathbb{R}^{2N}} \frac{\big(q_i(x) - q_i(y)\big)\big((q_j + \frac{\partial w}{\partial z_{\mu,\xi}} q_j)(x) - (q_j + \frac{\partial w}{\partial z_{\mu,\xi}} q_j)(y)\big)}{|x-y|^{N+2s}}\, dx\, dy \\
&= \sum_{i=1}^{N+1} \alpha_i \langle q_i, q_j \rangle + \sum_{i=1}^{N+1} \alpha_i \left\langle q_i, \frac{\partial w}{\partial z_{\mu,\xi}} q_j \right\rangle \qquad (4.2.61) \\
&= \lambda_j \alpha_j + \sum_{i=1}^{N+1} \alpha_i \left\langle q_i, \frac{\partial w}{\partial z_{\mu,\xi}} q_j \right\rangle,
\end{aligned}
$$

where Lemma 4.2.1 was also used in the last line.

Set now the $(N+1) \times (N+1)$ matrix $B^\varepsilon = (b^\varepsilon_{ij})$, defined as

$$
\begin{aligned}
b^\varepsilon_{ij} &:= \left\langle q_i, \frac{\partial w}{\partial \xi_j} \right\rangle, \quad i = 1, \dots N+1,\ j = 1, \dots, N, \\
b^\varepsilon_{i,N+1} &:= \left\langle q_i, \frac{\partial w}{\partial \mu} \right\rangle, \quad i = 1, \dots, N+1.
\end{aligned}
$$

By Cauchy-Schwartz inequality and (4.2.11) one has

$$
\lim_{\varepsilon \to 0} \left\langle q_i, \frac{\partial w}{\partial \xi_j} \right\rangle = \lim_{\varepsilon \to 0} \left\langle q_i, \frac{\partial w}{\partial \mu} \right\rangle = 0, \quad i = 1, \dots, N+1,\ j = 1, \dots, N, \tag{4.2.62}
$$

and thus $\lim_{\varepsilon \to 0} \|B^\varepsilon\| = 0$. Recalling that

$$
\frac{\partial w}{\partial z_{\mu,\xi}} q_j = \frac{\partial w}{\partial z_{\mu,\xi}} \frac{\partial z_{\mu,\xi}}{\partial \xi_j} = \frac{\partial}{\partial \xi_j} w(\varepsilon, z_{\mu,\xi}) = \frac{\partial w}{\partial \xi_j} \text{ for } j = 1, \dots, n
$$

and

$$
\frac{\partial w}{\partial z_{\mu,\xi}} q_{N+1} = \frac{\partial w}{\partial z_{\mu,\xi}} \frac{\partial z_{\mu,\xi}}{\partial \mu} = \frac{\partial}{\partial \mu} w(\varepsilon, z_{\mu,\xi}) = \frac{\partial w}{\partial \mu},
$$

equation (4.2.61) becomes

$$
\lambda_j \alpha_j + \sum_{i=1}^{N+1} \alpha_i b^\varepsilon_{ij} = 0, \quad i, j = 1, \dots, N+1,
$$

that is nothing but a $(N+1) \times (N+1)$ linear system with associated matrix $\lambda\, Id_{\mathbb{R}^{N+1}} + B^\varepsilon$, whose entries are $\lambda_j \delta_{ij} + b^\varepsilon_{ij}$, where $\delta_{jj} = 1$ and $\delta_{ij} = 0$ whether $i \neq j$.

Thus, since $\lim_{\varepsilon \to 0} \|B^\varepsilon\| = 0$, there exists $\varepsilon_1 > 0$ such that for $\varepsilon < \varepsilon_1$ the matrix $\lambda\, Id_{\mathbb{R}^{N+1}} + B^\varepsilon$ is invertible, and therefore $\alpha_i = 0$ for every $i = 1, \dots, N+1$. Hence, coming back to (4.2.60), we get

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{\big((z_{\mu,\xi} + w)(x) - (z_{\mu,\xi} + w)(y)\big)\big(\phi(x) - \phi(y)\big)}{|x-y|^{N+2s}}\, dx\, dy \\
&\qquad = \int_{\mathbb{R}^N} \Big(\varepsilon h(x) \big(z_{\mu,\xi}(x) + w(x)\big)^q + \big(z_{\mu,\xi}(x) + w(x)\big)^p \Big) \phi(x)\, dx,
\end{aligned}
$$

for every $\phi \in X^s$, that is, $z_{\mu,\xi} + w$ is a critical point of $\mathcal{I}_\varepsilon$. $\square$

4.3 Study of the behavior of Γ.

At this point, we have reduced our original problem to a finite-dimensional one. Indeed, we define the perturbed manifold

$$Z_\varepsilon := \{u := z_{\mu,\xi} + w(\varepsilon, z_{\mu,\xi}) \text{ s.t. } z_{\mu,\xi} \in Z_0\},$$

which is a natural constraint for the functional $\mathcal{I}_\varepsilon$.

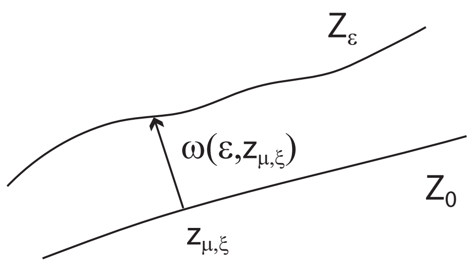


Figure 4.1: The perturbed manifold Z_ε.

We recall (4.1.7)and (4.2.2) and we give the following

Definition 4.3.1. We say that $u \in U$ is a proper local maximum (or minimum, respectively) of $\mathcal{I}$ if there exists a neighborhood $\mathcal{U}$ of u such that

$$\mathcal{I}(u) \geqslant \mathcal{I}(v) \ \ \forall\, v \in \mathcal{U} \qquad\qquad (\mathcal{I}(u) \leqslant \mathcal{I}(v) \ \ \forall\, v \in \mathcal{U}, \text{ respectively}),$$

and

$$\mathcal{I}(u) > \sup_{v\in\partial\mathcal{U}} \mathcal{I}(v) \qquad\qquad (\mathcal{I}(u) < \inf_{v\in\partial\mathcal{U}} \mathcal{I}(v), \text{ respectively}).$$

With this, one can prove that:

Proposition 4.3.2. *Suppose that $z_{\mu,\xi} \in Z_0$ is a proper local maximum or minimum of $\mathcal{I}$. Then, for $\varepsilon > 0$ sufficiently small, $u_\varepsilon := z_{\mu,\xi} + w(\varepsilon, z_{\mu,\xi}) \in Z_\varepsilon$ is a critical point of $\mathcal{I}_\varepsilon$.*

The proof of this can be found for instance in [22] (see in particular Theorem 2.16 there). A simple explanation goes as follows. First we notice that, for any $z_{\mu,\xi} \in Z_0$,

$$\mathcal{I}_0'(z_{\mu,\xi}) = 0, \tag{4.3.1}$$

where $\mathcal{I}_0$ is defined in (4.1.6). Indeed, $z_{\mu,\xi}$ is a critical point of $\mathcal{I}_0$, being a solution to (4.1.1). Now, recalling (4.1.5) and using Taylor expansion in the vicinity of $z_{\mu,\xi}$, we have

$$\begin{aligned}
\mathcal{I}_\varepsilon(z_{\mu,\xi} + w) &= \mathcal{I}_0(z_{\mu,\xi} + w) - \varepsilon\,\mathcal{I}(z_{\mu,\xi} + w) \\
&= \mathcal{I}_0(z_{\mu,\xi}) + \mathcal{I}_0'(z_{\mu,\xi})\, w + o(|w|) - \varepsilon\,\mathcal{I}(z_{\mu,\xi}) - \varepsilon\,\mathcal{I}'(z_{\mu,\xi})\, w + o(\varepsilon) \\
&= \mathcal{I}_0(z_{\mu,\xi}) - \varepsilon\,\mathcal{I}(z_{\mu,\xi}) + o(\varepsilon) \\
&= \mathcal{I}_0(z_0) - \varepsilon\,\mathcal{I}(z_{\mu,\xi}) + o(\varepsilon),
\end{aligned}$$

where we have used (4.3.1) and (4.2.11), and the translation and dilation invariance of $\mathcal{I}_0$ (z_0 was defined in (4.1.2)).

Therefore, we have reduced our problem to find critical points of $\mathcal{I}$. For this, we set

$$\Gamma(\mu,\xi) := \mathcal{I}(z_{\mu,\xi}) = \frac{\mu^{-\beta_s}}{q+1}\int_{\mathbb{R}^N} h(x) z_0^{q+1}\left(\frac{x-\xi}{\mu}\right)dx, \tag{4.3.2}$$

where

$$\beta_s := \frac{(N-2s)(q+1)}{2}. \tag{4.3.3}$$

Now we prove some lemmata concerning the behavior of Γ. In the first one we compute the limit of Γ as μ tends to zero.

Lemma 4.3.3. *Let* Γ *be as in* (4.3.2). *Then*

$$\lim_{\mu\to 0}\Gamma(\mu,\xi) = 0 \quad \textit{uniformly in } \xi.$$

Proof. Thanks to (4.0.2), there exists $r>1$ such that

$$\omega = supp\, h \subset B_r. \tag{4.3.4}$$

We first suppose that $\xi\in\mathbb{R}^N$ is such that $|\xi|\geqslant 2r$. Therefore, if $|y|<r$ then

$$|\xi+y|\geqslant|\xi|-|y|>r,$$

and so $y+\xi\in B_r^c\subset\omega^c$. This implies that

$$h(y+\xi)=0 \ \text{ if } |\xi|\geqslant 2r \text{ and } |y|<r. \tag{4.3.5}$$

Now, we observe that, using the change of variable $y=x-\xi$, Γ can be written as

$$\Gamma(\mu,\xi) = \frac{\mu^{-\beta_s}}{q+1}\int_{\mathbb{R}^N} h(y+\xi)\, z_0^{q+1}\left(\frac{y}{\mu}\right)dy.$$

Hence, using (4.3.5) we have that, if $|\xi|\geqslant 2r$,

$$\begin{aligned}\Gamma(\mu,\xi) &= \frac{\mu^{-\beta_s}}{q+1}\int_{|y|\geqslant r} h(y+\xi)\, z_0^{q+1}\left(\frac{y}{\mu}\right)dy\\ &\leqslant \frac{\mu^{-\beta_s}}{q+1}\max_{|y|\geqslant r} z_0^{q+1}\left(\frac{y}{\mu}\right)\int_{|y|\geqslant r} h(y+\xi)\,dy.\end{aligned}$$

This implies that

$$|\Gamma(\mu,\xi)|\leqslant\frac{\mu^{-\beta_s}}{q+1}\max_{|y|\geqslant r} z_0^{q+1}\left(\frac{y}{\mu}\right)\|h\|_{L^1(\mathbb{R}^N)}. \tag{4.3.6}$$

Now, recalling (4.1.2), we obtain that

$$z_0^{q+1}\left(\frac{y}{\mu}\right) = \alpha_{N,s}^{q+1} \frac{\mu^{(N-2s)(q+1)}}{(\mu^2+|y|^2)^{\frac{(N-2s)(q+1)}{2}}},$$

and so

$$\max_{|y|\geqslant r} z_0^{q+1}\left(\frac{y}{\mu}\right) = \mu^{(N-2s)(q+1)} \max_{|y|\geqslant r} \frac{\alpha_{N,s}^{q+1}}{(\mu^2+|y|^2)^{\frac{(N-2s)(q+1)}{2}}} \leqslant C\,\mu^{(N-2s)(q+1)},$$

for a suitable constant $C>0$ independent of μ. Using this in (4.3.6) and recalling (4.3.3), (4.0.2) and the fact that h is continuous, we get (up to renaming C)

$$|\Gamma(\mu,\xi)| \leqslant C\,\mu^{\frac{(N-2s)(q+1)}{2}},$$

which tends to zero as $\mu\to 0$. This concludes the proof in the case $|\xi|\geqslant 2r$.

If instead $|\xi|<2r$ then one has

$$\begin{aligned} \int_{\mathbb{R}^N} h(x)\, z_0^{q+1}\left(\frac{x-\xi}{\mu}\right) dx &= \int_{|x|<r} h(x)\, z_0^{q+1}\left(\frac{x-\xi}{\mu}\right) dx \\ &\leqslant \|h\|_{L^\infty(\mathbb{R}^N)} \int_{|x|<r} z_0^{q+1}\left(\frac{x-\xi}{\mu}\right) dx, \end{aligned} \tag{4.3.7}$$

thanks to (4.3.4) and the fact that h is continuous.

We claim that

$$\int_{|x|<r} z_0^{q+1}\left(\frac{x-\xi}{\mu}\right) dx \leqslant C\,\mu^{\min\{N,(N-2s)(q+1)\}}, \tag{4.3.8}$$

for some positive constant C independent of μ (possibly depending on r). To prove this, we recall (4.1.2) and we get

$$\begin{aligned} \int_{|x|<r} z_0^{q+1}\left(\frac{x-\xi}{\mu}\right) dx &= \alpha_{N,s}^{q+1} \int_{|x|<r} \frac{dx}{\left(1+\frac{|x-\xi|^2}{\mu^2}\right)^{\frac{(N-2s)(q+1)}{2}}} \\ &\leqslant \alpha_{N,s}^{q+1}\left(\int_{|x-\xi|\leqslant\mu} dx + \int_{\mu<|x-\xi|<3r} \frac{\mu^{(N-2s)(q+1)}}{|x-\xi|^{(N-2s)(q+1)}}\,dx\right) \\ &\leqslant C\left(\mu^N + \mu^{(N-2s)(q+1)} \int_\mu^{3r} \rho^{N-1-(N-2s)(q+1)}\,d\rho\right) \\ &\leqslant C\left(\mu^N + \mu^{(N-2s)(q+1)}\,\mu^{-[(N-2s)(q+1)-N]_+}\right) \\ &\leqslant C\left(\mu^N + \mu^{\min\{N,(N-2s)(q+1)\}}\right) \\ &\leqslant C\,\mu^{\min\{N,(N-2s)(q+1)\}}, \end{aligned}$$

up to changing C from line to line, and this shows (4.3.8). Therefore, by (4.3.2), (4.3.3) and (4.3.7) we have that

$$|\Gamma(\mu,\xi)| \leqslant C\,\mu^{-\frac{(N-2s)(q+1)}{2}}\,\mu^{\min\{N,(N-2s)(q+1)\}}.$$

Hence, if $(N-2s)(q+1) \leqslant N$ we get that

$$|\Gamma(\mu,\xi)| \leqslant C\,\mu^{(N-2s)(q+1)},$$

which implies that $\Gamma(\mu,\xi)$ tends to zero as $\mu \to 0$. If instead $N < (N-2s)(q+1)$ we obtain that

$$|\Gamma(\mu,\xi)| \leqslant C\,\mu^{N-\frac{(N-2s)(q+1)}{2}}.$$

In this case, we observe that, since $q \in (0,p)$ with $p = \frac{N+2s}{N-2s}$, then $q+1 < \frac{2N}{N-2s}$, and so

$$N - \frac{(N-2s)(q+1)}{2} > N - \frac{N-2s}{2}\,\frac{2N}{N-2s} = 0.$$

This implies that also in this case $\Gamma(\mu,\xi)$ tends to zero as $\mu \to 0$. This concludes the proof of Lemma 4.3.3. ☐

Now we compute the limit of Γ as $\mu + |\xi|$ tends to $+\infty$.

Lemma 4.3.4. *Let* Γ *be as in* (4.3.2)*. Then*

$$\lim_{\mu+|\xi|\to+\infty} \Gamma(\mu,\xi) = 0.$$

Proof. Suppose that $\mu \to +\infty$. Then recalling (4.0.2), the fact that h is continuous and (4.1.2) we have

$$|\Gamma(\mu,\xi)| \leqslant C\,\mu^{-\beta_s}\,\|h\|_{L^1(\mathbb{R}^N)} \leqslant C\mu^{-\beta_s},$$

for some positive constant C, changing at every inequality, independent of μ. Therefore $\Gamma(\mu,\xi)$ tends to zero as $\mu \to +\infty$.

Suppose now that $\mu \to \bar{\mu}$ for some $\bar{\mu} \in [0,+\infty)$, therefore $|\xi| \to +\infty$. If $\bar{\mu} = 0$, then we can use Lemma 4.3.3 and we get the desired result. Hence, we can suppose that $\bar{\mu} \in (0,+\infty)$. In this case, we make the change of variable $y = x - \xi$ and we write Γ as

$$\Gamma(\mu,\xi) = \frac{\mu^{-\beta_s}}{q+1}\int_{\mathbb{R}^N} h(y+\xi)\,z_0^{q+1}\left(\frac{y}{\mu}\right)\,dy. \tag{4.3.9}$$

Since h has compact support (recall (4.0.2)), there exists $r > 0$ such that $\omega = supp\, h \subset B_r$ and so (4.3.9) becomes

$$\Gamma(\mu,\xi) = \frac{\mu^{-\beta_s}}{q+1}\int_{|y+\xi|\leqslant r} h(y+\xi)\,z_0^{q+1}\left(\frac{y}{\mu}\right)\,dy. \tag{4.3.10}$$

We also notice that, since $|\xi| \to +\infty$, we can suppose that $|\xi| > 2r$. Therefore, if we consider $y \in B_r(-\xi)$, then $|y+\xi| \leqslant r < |\xi|/2$, which implies that

$$|y| \geqslant |\xi| - |y+\xi| \geqslant |\xi| - \frac{|\xi|}{2} = \frac{|\xi|}{2}.$$

Hence, recalling (4.1.2), we obtain that if $y \in B_r(-\xi)$

$$\begin{aligned} z_0^{q+1}\left(\frac{y}{\mu}\right) &= \frac{\alpha_{N,s}^{q+1}\,\mu^{(N-2s)(q+1)}}{(\mu^2+|y|^2)^{\frac{(N-2s)(q+1)}{2}}} \\ &\leqslant \frac{\alpha_{N,s}^{q+1}\,\mu^{(N-2s)(q+1)}}{|y|^{(N-2s)(q+1)}} \\ &\leqslant \frac{2^{(N-2s)(q+1)}\,\alpha_{N,s}^{q+1}\,\mu^{(N-2s)(q+1)}}{|\xi|^{(N-2s)(q+1)}}. \end{aligned}$$

Using this, (4.0.2) and the fact that h is continuous into (4.3.10), we have that

$$|\Gamma(\mu,\xi)| \leqslant C\,\mu^{\beta_s}\,\frac{1}{|\xi|^{(N-2s)(q+1)}}\,\|h\|_{L^1(\mathbb{R}^N)} \leqslant C\frac{\mu^{\beta_s}}{|\xi|^{(N-2s)(q+1)}},$$

for some constant independent of μ and ξ. Since $\mu \to \bar{\mu} \in (0,+\infty)$, this implies that

$$\Gamma(\mu,\xi) \to 0 \quad \text{as } |\xi| \to +\infty,$$

concluding the proof of Lemma 4.3.4. □

Finally we show the following:

Lemma 4.3.5. *Let* Γ *be as in* (4.3.2). *Suppose that there exists* $\xi_0 \in \mathbb{R}^N$ *such that* $h(\xi_0) > 0$ *(*$h(\xi_0) < 0$ *respectively). Then*

$$\lim_{\mu\to 0}\frac{\Gamma(\mu,\xi_0)}{\mu^{N-\beta_s}} = A,$$

for some $A > 0$*, possibly* $A = +\infty$ *(*$A < 0$*, possibly* $A = -\infty$*, respectively).*

Proof. We prove the lemma only in the case $h(\xi_0) > 0$, since the other case is analogous. We notice that, by using the change of variable $y = (x-\xi)/\mu$, we can rewrite Γ as

$$\Gamma(\mu,\xi) = \frac{\mu^{N-\beta_s}}{q+1}\int_{\mathbb{R}^N} h(\mu y+\xi)\, z_0^{q+1}(y)\,dy. \tag{4.3.11}$$

Then, we obtain

$$\frac{\Gamma(\mu,\xi_0)}{\mu^{N-\beta_s}} = \frac{1}{q+1}\int_{\mathbb{R}^N} h(\mu y+\xi_0)\, z_0^{q+1}(y)\,dy,$$

and we observe that

$$h(\mu y+\xi_0)\, z_0^{q+1}(y) \to h(\xi_0)\, z_0^{q+1}(y) \quad \text{as } \mu \to 0.$$

Suppose first that $\frac{2s}{N-2s} < q < p$. In this case, we have that z_0 defined in (4.1.2) satisfies

$$z_0^{q+1} \in L^1(\mathbb{R}^N).$$

Thus, thanks to (4.0.2) and the fact that h is continuous, we have that

$$h(\mu y + \xi_0)\, z_0^{q+1}(y) \leqslant \|h\|_{L^\infty(\mathbb{R}^N)}\, z_0^{q+1}(y) \in L^1(\mathbb{R}^N),$$

and so from the Dominated Convergence Theorem, we get

$$\frac{\Gamma(\mu,\xi_0)}{\mu^{N-\beta_s}} \to \frac{h(\xi_0)}{q+1} \int_{\mathbb{R}^N} z_0^{q+1}(y)\, dy \quad \text{as } \mu \to 0,$$

showing the lemma in the case $\frac{2s}{N-2s} < q < p$. Notice that in this case

$$A := \frac{h(\xi_0)}{q+1} \int_{\mathbb{R}^N} z_0^{q+1}(y)\, dy$$

is strictly positive and bounded.

If instead $z_0^{q+1} \notin L^1(\mathbb{R}^N)$, then we use Fatou's Lemma to get

$$\liminf_{\mu\to 0} \int_{\mathbb{R}^N} h(\mu y + \xi_0)\, z_0^{q+1}(y)\, dy \geqslant h(\xi_0) \int_{\mathbb{R}^N} z_0^{q+1}(y)\, dy,$$

which implies that in this case $A := +\infty$. This concludes the proof of Lemma 4.3.5. □

4.4 Solvability of the problem. Multiplicity.

Now we are ready to complete the proof of Theorem 4.1.2.

We observe that, thanks to (4.0.3) and Lemma 4.3.5, there exist $\mu_0 > 0$ as small as we want and $\xi_0 \in \mathbb{R}^N$ such that

$$\Gamma(\mu_0,\xi_0) \geqslant \frac{\mu_0^{N-\beta_s}}{2} \min\{A, 1\} =: B. \tag{4.4.1}$$

Now, we use Lemma 4.3.3 to say that if $\mu > 0$ is sufficiently small, then

$$\Gamma(\mu,\xi) < \frac{B}{2} \text{ for any } \xi \in \mathbb{R}^N.$$

In particular, if $\mu_1 := \mu_0/2$, then

$$\Gamma(\mu_1,\xi) < \frac{B}{2} \text{ for any } \xi \in \mathbb{R}^N. \tag{4.4.2}$$

Moreover, from Lemma 4.3.4 we deduce that there exists $R_* > 0$ such that if $\mu + |\xi| > R_*$ we have

$$\Gamma(\mu,\xi) < \frac{B}{2}.$$

In particular, we can take $\mu_2 = R_2 = R_* + \mu_0 + |\xi_0| + 1$ and we have that

$$\Gamma(\mu,\xi) < \frac{B}{2} \text{ if either } \mu = \mu_2 \text{ and } |\xi| \leqslant R_2 \text{ or } \mu \leqslant \mu_2 \text{ and } |\xi| = R_2. \tag{4.4.3}$$

Now we perform our choice of R, μ_1 and μ_2 in (4.2.1): we take μ_1 and μ_2 such that (4.4.2) and (4.4.3) are satisfied, and $R = R_2$.

Also, we set

$$S := \{\mu_1 \leqslant \mu \leqslant \mu_2 \text{ and } |\xi| \leqslant R\},$$

and we notice that Γ admits a maximum in S, since Γ is continuous and S is a compact set. Moreover, thanks to (4.4.2) and (4.4.3) we have that

$$\Gamma(\mu, \xi) < \frac{B}{2} \text{ if } (\mu, \xi) \in \partial S. \tag{4.4.4}$$

On the other hand,

$$|\xi_0| < R_2 \text{ and } \mu_1 < \mu_0 < \mu_2,$$

which implies that $(\mu_0, \xi_0) \in S$. Therefore, (4.4.1) and (4.4.4) imply that the maximum of Γ is achieved at some point (μ_*, ξ_*) in the interior of S.

Now, we go back to the functional $\mathcal{I}$, and recalling (4.3.2) we obtain that $\mathcal{I}$ admits a maximum z_{μ_*,ξ_*} in the critical manifold Z_0 defined in (4.2.1). Hence, we can apply Proposition 4.3.2 and we obtain the existence of a critical point of $\mathcal{I}_\varepsilon$, that is a solution to (4.0.1), given by

$$u_{1,\varepsilon} := z_{\mu_*,\xi_*} + w(\varepsilon, z_{\mu_*,\xi_*}).$$

Also, $u_{1,\varepsilon}$ is positive and tends to z_{μ_*,ξ_*} in X^s as $\varepsilon \to 0$ thanks to (4.2.11).

Furthermore, if h changes sign, then there exists $\tilde{\xi}_0 \in \mathbb{R}^N$ such that $h(\tilde{\xi}_0) < 0$, and so we can use Lemma 4.3.5 to say that

$$\Gamma(\tilde{\mu}_0, \tilde{\xi}_0) \leqslant \frac{\tilde{\mu}_0^{N-\beta_s}}{2} \max\{A, -1\},$$

for some $\tilde{\mu}_0 > 0$. Then we can repeat all the arguments above (with suitable modifications) to find a local minimum of Γ, and so a a local minimum of $\mathcal{I}$. Then, again from Proposition 4.3.2 we obtain the existence of a second positive solution. This concludes the proof of Theorem 4.1.2.

PART III

Biharmonic elliptic problems

Chapter 5

A biharmonic problem involving the 2-Hessian operator

Finally, in this last chapter, we leave the nonlocal framework and we focus on a local problem, leaded by a 4-th order operator. In particular, we will consider the following model problem,

$$\begin{cases} \Delta^2 u = S_2(D^2 u), & \text{in } \Omega \subset \mathbb{R}^N, \\ B(u) = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.0.1}$$

where $S_2(D^2u)(x) = \sum_{1 \leqslant i < j \leqslant N} \lambda_i(x)\lambda_j(x)$, being λ_i, $i = 1, \cdots, N$, the solutions to the equation

$$\det\left(\lambda I - D^2 u(x)\right) = 0,$$

Δ^2 is the bi-Laplacian operator and $N = 3$. By $B(u)$ we mean some generic boundary conditions (Dirichlet and Navier in this work) that will be specified when needed.

It is worth pointing out here that the case $N = 2$ appears as the stationary part of a model of epitaxial growth of crystals (see [90, 140]) initially studied in [91]. In dimension $N = 3$ the model can be seen as the stationary part of a 3-dimensional growth problem driven by the scalar curvature.

To make a deep study of this problem, and of the questions arising from it, we will lean on the functional background already analyzed in the Introduction. Indeed, as we announced there, these 4-th order problems come strongly determined by the type of boundary conditions they satisfy, for instance in what concerns to their variational structure.

More precisely, we will find natural restrictions in our analysis mainly linked to the following subjects.

- The deep dependence on the boundary conditions of the variational formulation of the nonlinear term $S_2(D^2u)$.
- The restriction on the dimension N to use the critical point theory (determined by the lack of compactness at the gradient level).

- The necessity of applying alternative techniques when the variational formulation is not valid (Navier conditions), that can also depend on some reaction and source terms.

As third point suggests, it will be natural, both from the theoretical and applied point of view, to study the effect of a source term. Indeed, this term describes, roughly speaking, the amount of material provided to the system from the exterior, and moreover, takes into account a local reaction term in the behavior of the problem. In the case of Dirichlet boundary conditions a reaction term influences the multiplicity of nontrivial solutions. Moreover, sometimes, it changes the stability of the equilibrium to the zero solution. In the case of Navier boundary conditions, it is worth to point out that, when the problem does not admit a variational formulation, a suitable reaction term provides a nontrivial solution.

From now on, when nothing is specified, we are supposing $N = 3$, that is, Ω is a smooth domain of $\mathbb{R}^3$. Finally, we precise the concept of solutions that we will use along this chapter. Let $F(x, u) \in L^1(\Omega)$, and consider the general problem

$$\begin{cases} \Delta^2 u = F(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.0.2}$$

where Dirichlet boundary conditions are prescribed. In this case, we will refer to solutions in the next sense.

Definition 5.0.1. We say that $u \in W_0^{2,2}(\Omega)$ is an (energy) solution to (5.0.2) if

$$\int_\Omega \Delta u \Delta \varphi \, dx = \int_\Omega F(x, u)\varphi(x) \, dx,$$

for every $\varphi \in W_0^{2,2}(\Omega)$.

Likewise, when we deal with a problem with Navier boundary conditions, that is,

$$\begin{cases} \Delta^2 u = F(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.0.3}$$

by solutions we will understand the following.

Definition 5.0.2. We say that $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is an (energy) solution to (5.0.3) if

$$\int_\Omega \Delta u \Delta \varphi \, dx = \int_\Omega F(x, u)\varphi(x) \, dx,$$

for every $\varphi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

Remark 5.0.3. *Since $N = 3$, by Theorem 0.0.8, every $\varphi \in W^{2,2}(\Omega)$ belongs in particular to $L^\infty(\Omega)$. Thus, the integrals in the previous definitions are well defined by asking only $F(x, u) \in L^1(\Omega)$.*

5.1 The 2-Hessian: Functional setting and variational formulation.

This section is devoted to analyze, when it is possible, the variational formulation of the problem (5.0.1). With this purpose, we expose first several results concerning the regularity of the nonlinear term $S_2(D^2u)$.

Indeed, let S_2 be the 2-Hessian operator. Namely, we set

$$S_2(D^2u(x)) := \sum_{1\leqslant i<j\leqslant N} \lambda_i(x)\lambda_j(x),$$

where $\lambda_i(x)$ $i = 1, \dots, N$ is the $i-th$ eigenvalue of the symmetric matrix $D^2u(x)$. For further details about $k-$Hessian operators, see [176] and, for some applications, [98, 99, 101].

Furthermore, we remind that S_2 can be also written in the following way,

$$S_2(D^2u(x)) = \sum_{1\leqslant i<j\leqslant N} \det(D^2_{ij}u(x)),$$

where

$$\det(D^2_{ij}u(x)) = \partial_{ii}u(x)\partial_{jj}u(x) - (\partial_{ij}u(x))^2.$$

Remark 5.1.1. *Notice that, for $u \in W^{2,2}(\Omega)$, $S_2(D^2u)$ is integrable in Ω.*

On the other hand, we remark that if u is a conveniently smooth function, then there holds

$$\begin{aligned}
&\partial_{ij}(\partial_i u\partial_j u) - \frac{1}{2}\partial_{ii}((\partial_j u)^2) - \frac{1}{2}\partial_{jj}((\partial_i u)^2) \\
(5.1.1)\qquad &\qquad = \partial_i(\partial_{ji}u\partial_j u + \partial_i u\partial_{jj}u) - \partial_i(\partial_j u\partial_{ij}u) - \partial_j(\partial_i u\partial_{ji}u) \\
&\qquad = \partial_{ii}u\partial_{jj}u - \partial_{ij}u\partial_{ij}u = \partial_{ii}u\partial_{jj}u - (\partial_{ij}u)^2.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
S_2(D^2u(x)) &= \sum_{1\leqslant i<j\leqslant N} \left(\partial_{ii}u(x)\partial_{jj}u(x) - (\partial_{ij}u(x))^2\right) \\
(5.1.2)\qquad &= \sum_{1\leqslant i<j\leqslant N} \left(\partial_{ij}(\partial_i u\partial_j u) - \frac{1}{2}\partial_{ii}((\partial_j u)^2) - \frac{1}{2}\partial_{jj}((\partial_i u)^2)\right).
\end{aligned}$$

Moreover,

$$(\Delta u(x))^2 = \left(\sum_{i=1}^{N} \partial_{ii}u(x)\right)^2 = \sum_{i=1}^{N}(\partial_{ii}u(x))^2 + 2\sum_{1\leqslant i<j\leqslant N} \partial_{ii}u(x)\partial_{jj}u(x),$$

i.e.,

$$(\Delta u(x))^2 - \sum_{i=1}^{N}(\partial_{ii}u(x))^2 = 2\sum_{1\leqslant i<j\leqslant N} \partial_{ii}u(x)\partial_{jj}u(x),$$

and

$$\begin{aligned}(\Delta u(x))^2 - \sum_{i=1}^{N}(\partial_{ii}u(x))^2 - 2\sum_{1\leqslant i<j\leqslant N}(\partial_{ij}u(x))^2 \\ = 2\sum_{1\leqslant i<j\leqslant N}\partial_{ii}u(x)\partial_{jj}u(x) - 2\sum_{1\leqslant i<j\leqslant N}(\partial_{ij}u(x))^2.\end{aligned}$$

Thus,

$$\begin{aligned}(\Delta u(x))^2 - \|D^2u(x)\|_2^2 &= (\Delta u(x))^2 - \sum_{i,j=1}^{N}(\partial_{ij}u(x))^2 \\ &= 2\sum_{1\leqslant i<j\leqslant N}(\partial_{ii}u(x)\partial_{jj}u(x) - (\partial_{ij}u(x))^2) \\ &= 2\sum_{1\leqslant i<j\leqslant N}\det(D^2_{ij}u(x)) \\ &= 2S_2(D^2u(x)).\end{aligned}$$

Moreover, for a better understanding of the 2-Hessian, we recall the definition of the Hardy space $\mathcal{H}^1(\mathbb{R}^N)$. Let us consider first R_j, the classical Riesz transform, defined as

$$R_j := \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}},\ j = 1, 2, \cdots, N,$$

or equivalently,

$$R_j := \mathcal{F}^{-1}i\frac{x_j}{|x|}\mathcal{F},\ j = 1, 2, \cdots, N.$$

Thus,

Definition 5.1.2. The Hardy space $\mathcal{H}^1(\mathbb{R}^N)$ is defined as

$$\mathcal{H}^1(\mathbb{R}^N) := \{f \in L^1(\mathbb{R}^N) \mid R_j(f) \in L^1(\mathbb{R}^N),\ j = 1, 2, \cdots, N\}$$

Furthermore, if we consider

$$h_t(x) := \frac{1}{t^N}\, h\left(\frac{x}{t}\right),\ \text{ where } h \in \mathcal{C}_0^\infty(\mathbb{R}^N),\ h(x) \geqslant 0 \text{ and } \int_{\mathbb{R}^N} h\, dx = 1,$$

it can be proved that indeed

$$\mathcal{H}^1(\mathbb{R}^N) = \{f \in L^1(\mathbb{R}^N) \mid \sup |f * h_t(x)| \in L^1(\mathbb{R}^N)\}.$$

See [163, 165] for further details. The following particular case of a result by R. Coifman, P.L. Lions, Y. Meyer and S. Semmes (see [70]) gives distributional sense to the identities above for functions in $W^{2,2}(\mathbb{R}^N)$.

Lemma 5.1.3. *(R. Coifman, P.L. Lions, Y. Meyer and S. Semmes). Let U, V vector fields in $\mathbb{R}^N$ such that ∇U, $\nabla V \in \left[L^2(\mathbb{R}^N)\right]^{N\times N}$ and $div(U) = div(V) = 0$ in $\mathcal{D}'(\mathbb{R}^N)$. Then*

$$\sum_{i,j=1}^{N} \partial_{ij}(U_i V_j) \in \mathcal{H}^1(\mathbb{R}^N),$$

where U_i and V_j denote the i-th and j-th components of U and V respectively.

The proof of Lemma 5.1.3 involves some techniques from Harmonic Analysis and an adaptation of the ideas by Luc Tartar on *compensation compactness*. See, for the last subject, the references [171, 172]. We will use a localization of the following result, which is a Corollary of Lemma 5.1.3.

Lemma 5.1.4. *If $u \in W^{2,2}(\mathbb{R}^N)$, $N \geqslant 2$, then*

$$\sum_{i\neq j}\left(\partial_{ij}(\partial_i u \partial_j u) - \frac{1}{2}\partial_{ii}((\partial_j u)^2) - \frac{1}{2}\partial_{jj}((\partial_i u)^2)\right) \in \mathcal{H}^1(\mathbb{R}^N).$$

Proof. For any $1 \leqslant i < j \leqslant N$ we define

$$U^{i,j} := (U_1^{i,j}, U_2^{i,j}, \ldots, U_N^{i,j})$$

where $U_k^{i,j} = 0$, if $k \neq i, j$ and $U_i^{i,j} = -\partial_j u$, $U_j^{i,j} = \partial_i u$. In particular

$$\mathrm{div}(U^{i,j}) = 0.$$

Now let us denote $U = U^{i,j}$ and $V = U^{i,j}$. Then $U_i V_i = (\partial_j u)^2$, $U_j V_j = (\partial_i u)^2$, and $U_i V_j = U_j V_i = -\,\partial_j u \partial_i u$, otherwise $U_k V_l = 0$, whenever $k \neq i, j$ or $l \neq i, j$. In particular recalling Lemma 5.1.3 it yields

$$-\frac{1}{2}\sum_{k,l=1}^{N}\partial_{kl}(U_k V_l) = \partial_{ij}(\partial_i u \partial_j u) - \frac{1}{2}\partial_{ii}((\partial_j u)^2) - \frac{1}{2}\partial_{jj}((\partial_i u)^2) \in \mathcal{H}^1(\mathbb{R}^N).$$

By linearity, we conclude

$$\sum_{i<j}\left(\partial_{ij}(\partial_i u \partial_j u) - \frac{1}{2}\partial_{ii}((\partial_j u)^2) - \frac{1}{2}\partial_{jj}((\partial_i u)^2)\right) \in \mathcal{H}^1(\mathbb{R}^N)$$

□

Remark 5.1.5. *Since in the study of the problems with Dirichlet boundary conditions we work in $W_0^{2,2}(\Omega)$, if $\partial\Omega$ is smooth we can extend the function by zero outside Ω to apply Lemma 5.1.4.*

Consequently, if we consider $u, v \in \mathcal{C}_0^\infty(\Omega)$, integrating by parts we obtain

(5.1.3)

$$\begin{aligned}
&\int_\Omega (\Delta^2 u - S_2(D^2 u))v \, dx \\
&\quad = \int_\Omega \Delta u \Delta v \, dx - \int_\Omega \sum_{1 \leqslant i < j \leqslant N} \Big((\partial_i u \partial_j u) \partial_{ij} v + \frac{1}{2} \partial_i (\partial_j u)^2 \partial_i v + \frac{1}{2} \partial_j (\partial_i u)^2 \partial_j v \Big) \, dx \\
&\quad = \int_\Omega \Delta u \Delta v \, dx - \int_\Omega \sum_{1 \leqslant i < j \leqslant N} \Big(\partial_i u \partial_j u \partial_{ij} v + \partial_j u \partial_{ij} u \partial_i v + \partial_i u \partial_{ji} u \partial_j v \Big) \, dx.
\end{aligned}$$

Thus, if we define the energy functional $\mathcal{E} : W_0^{2,2}(\Omega) \to \mathbb{R}$ as

$$\mathcal{E}(u) := \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - \int_\Omega \sum_{1 \leqslant i < j \leqslant N} \partial_{ij} u \partial_i u \partial_j u \, dx,$$

then

$$\frac{d\mathcal{E}(u + \varepsilon v)}{d\varepsilon}\bigg|_{\varepsilon = 0} = \int_\Omega \Delta u \Delta v - \int_\Omega \sum_{1 \leqslant i < j \leqslant N} \Big(\partial_i u \partial_j u \partial_{ij} v + \partial_j u \partial_{ij} u \partial_i v + \partial_i u \partial_{ji} u \partial_j v \Big),$$

that is, the Dirichlet problem is the Euler-Lagrange equation of $\mathcal{E}$.

We can summarize the previous computation in the following result.

Proposition 5.1.6. *If $u \in W_0^{2,2}(\Omega)$ is a critical point of $\mathcal{E}$, then it is an energy solution of the Dirichlet problem*

$$\begin{cases} \Delta^2 u = S_2(D^2 u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 5.1.7. *It is easy to check that if we consider Navier boundary conditions, that is, prescribing $u = \Delta u = 0$ on the boundary, and we repeat the computations in (5.1.3) for $v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, one finds some boundary integrals which do not vanish. Thus, Navier conditions do not admit a Lagrangian and we cannot study the problem in a variational formulation.*

5.2 The Dirichlet problem.

Consider the problem

$$\begin{cases} \Delta^2 u = S_2(D^2 u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2.1}$$

where Ω is a smooth domain of $\mathbb{R}^3$. By Proposition 5.1.6, we know that critical points of $\mathcal{E}$ are solutions of (5.2.1), and therefore, to study the solvability of this problem, we will analyze the behavior of the energy functional. Actually, despite of the presence of the partial derivatives, looking at the form of the functional one can think of its second term,

$$\int_\Omega \sum_{1\leqslant i<j\leqslant N} \partial_{ij}u\partial_i u\partial_j u\,dx,$$

as a term of homogeneity order 3 in u. That is, roughly speaking, the functional would be made up of a term of order 2, with positive sign, and a term of order 3, with indefinite sign. Thus, attending to the variational techniques developed along Chapter 1, it seems reasonable to expect the Mountain Pass Lemma to provide a solution of (5.2.1).

Indeed, we can prove that $\mathcal{E}$ satisfies the Palais-Smale condition.

Lemma 5.2.1. *Assume $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{2,2}(\Omega)$ is a bounded Palais-Smale sequence for $\mathcal{E}$, that is, $\{u_n\}_{n\in\mathbb{N}}$ verifies*

(i) $\|\Delta u_n\|_{L^2(\Omega)} \leqslant C$, *with $C>0$ independent of n,*

(ii) $\mathcal{E}(u_n)\to c$ *as $n\to\infty$,*

(iii) $\mathcal{E}'(u_n)\to 0$ *in $W^{-2,2}(\Omega)$.*

Then there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ that converges in $W_0^{2,2}(\Omega)$.

Proof. Since $\|\Delta u_n\|_{L^2(\Omega)} \leqslant C$, up to a subsequence, by the Rellich-Kondrachov Theorem for the space $W_0^{2,2}(\Omega)$ (see for example [114, Theorem 2.4]) we know that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } W_0^{2,2}(\Omega),\\ &u_n \to u \text{ in } L^p(\Omega), \text{ for all } 1\leqslant p<\infty,\\ &u_n\to u \text{ a.e. in } \Omega.\end{aligned} \tag{5.2.2}$$

Moreover, $\|\nabla u_n\|_{W_0^{1,2}(\Omega)} \leqslant C$. Hence, applying now the Rellich-Kondrachov Theorem for $W_0^{1,2}(\Omega)$ we also obtain that

$$\nabla u_n \to \nabla u \text{ in } [L^q(\Omega)]^N, \quad 1\leqslant q<6. \tag{5.2.3}$$

On the other hand, we can write condition (iii) as

$$\Delta^2 u_n = S_2(D^2u_n) + y_n,\ u_n\in W_0^{2,2}(\Omega) \text{ and } y_n\to 0 \text{ in } W^{-2,2}(\Omega),$$

and multiplying here by (u_n-u), we have

$$\int_\Omega \Delta u_n \Delta(u_n-u)\,dx = \int_\Omega (u_n-u)S_2(D^2u_n)\,dx + \int_\Omega y_n(u_n-u)\,dx. \tag{5.2.4}$$

The last term on the right hand side goes to zero due to the convergence of y_n. For the first one, using (5.1.2) and integrating by parts, it yields

$$\begin{aligned}\int_\Omega (u_n-u)S_2(D^2u_n)\,dx &= \int_\Omega \sum_{1\leqslant i<j\leqslant N} \Big(-\partial_{ii}u_n\partial_j u_n\partial_j(u_n-u) - \partial_i u_n\partial_{ij}u_n\partial_j(u_n-u)\\ &\qquad + \partial_j u_n\partial_{ij}u_n\partial_i(u_n-u) + \partial_i u_n\partial_{ji}u_n\partial_j(u_n-u)\Big)\,dx\\ &= \int_\Omega \sum_{1\leqslant i<j\leqslant N} \Big(-\partial_{ii}u_n\partial_j u_n\partial_j(u_n-u) + \partial_j u_n\partial_{ij}u_n\partial_i(u_n-u)\Big)\,dx.\end{aligned}$$

Using now Hölder's inequality, the uniform boundedness of u_n in $W_0^{2,2}(\Omega)$ and the convergences in (5.2.2) and (5.2.3), there holds

$$\begin{aligned}\int_\Omega (u_n-u)S_2(D^2u_n)\,dx &\leqslant C\int_\Omega |D^2u_n||\nabla u_n||\nabla(u_n-u)|\,dx\\ &\leqslant C\|\Delta u_n\|_{L^2(\Omega)}\|\nabla u_n\|_{L^4(\Omega)}\|\nabla u_n-\nabla u\|_{L^4(\Omega)}\to 0.\end{aligned}$$

Therefore, from (5.2.4) we conclude

$$\int_\Omega \Delta u_n\Delta(u_n-u)\,dx\to 0 \quad\text{when}\quad n\to\infty.$$

Otherwise, by the weak convergence in $W_0^{2,2}(\Omega)$,

$$\int_\Omega \Delta u\Delta(u_n-u)\,dx\to 0 \text{ when } n\to\infty,$$

and hence, substracting these two terms we reach

$$\int_\Omega |\Delta(u_n-u)|^2\,dx\to 0,$$

so that Palais-Smale condition is satisfied. □

Remark 5.2.2. *This result does not hold for $N=4$. The convergence of the gradients in $L^4(\Omega)$ cannot be obtained by the Rellich-Kondrachov Theorem, because the critical exponent at the gradient level is*

$$\frac{2N}{N-2}=4.$$

Hence, only with this approach we cannot pass to the limit in (5.2)*.*

On the other hand, just by applying Hölder and Sobolev's inequalities we can obtain information about the geometry of the functional. In fact,

$$\begin{aligned}\mathcal{E}(u) &\geqslant \frac12\int_\Omega |\Delta u|^2\,dx - C\left(\int_\Omega |\Delta u|^2\,dx\right)^{\frac12}\left(\int_\Omega |\nabla u|^4\,dx\right)^{\frac12}\\ &\geqslant \frac12\int_\Omega |\Delta u|^2\,dx - c_1\left(\int_\Omega |\Delta u|^2\,dx\right)^{\frac32}\\ &= \frac12\|\Delta u\|^2_{L^2(\Omega)} - c_1\|\Delta u\|^3_{L^2(\Omega)} = h\left(\|\Delta u\|_{L^2(\Omega)}\right),\end{aligned}$$

where

$$h(s) := \frac{1}{2}s^2 - c_1 s^3, \quad s > 0. \tag{5.2.5}$$

Notice that here it is clear what we pointed out at the begining of the section, the behavior of the non linear term as a convex power in the functional. Thus, we can already prove the existence result of this section.

Theorem 5.2.3. *Problem* (5.2.1) *has at least one nontrivial solution.*

Proof. We proceed in several steps.

Step 1: First of all, it can be checked the existence of a function $\psi \in W_0^{2,2}(\Omega)$ such that

$$\int_\Omega \sum_{1\leqslant i<j\leqslant N} \partial_{ij}\psi \partial_i\psi \partial_j\psi \, dx > 0.$$

See for instance [91], where the authors provide an specific function satisfying this condition for $N = 2$. As a consequence, $\mathcal{E}(s\psi) < 0$ for s large enough. Moreover, it trivially follows that the function h, defined in (5.2.5), has a local positive maximum.

Step 2: $\mathcal{E}$ satisfies the Mountain Pass geometry.
This claim is obtained as a straigthforward application of Step 1. Indeed, from this information we know that there exist α, $\beta > 0$ such that:

(a) $\mathcal{E}(u) \geqslant \beta$ for every $u \in W_0^{2,2}(\Omega)$ with $\|\Delta u\|_{L^2(\Omega)} = \alpha$.

(b) There exists $v \in W_0^{2,2}(\Omega)$ with $\|\Delta v\|_{L^2(\Omega)} > \alpha$ and $\mathcal{E}(v) < \beta$.

Therefore, $\mathcal{E}$ satisfies the Mountain Pass geometry. Define now

$$\Gamma := \{\gamma \in \mathcal{C}\left([0,1], W_0^{2,2}(\Omega)\right) \mid \gamma(0) = 0,\, \gamma(1) = v\},$$

and the mini-max value

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{E}[\gamma(t)].$$

By the Ekeland variational principle (see [87]), there exists a Palais-Smale sequence at level c, i.e., there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{2,2}(\Omega)$ such that

(i) $\mathcal{E}(u_n) \to c$,

(ii) $\mathcal{E}'(u_n) \to 0$ in $W^{-2,2}(\Omega)$,

as $n \to \infty$.

Step 3: Every Palais-Smale sequence is uniformly bounded in $W_0^{2,2}(\Omega)$.
If $u \in W_0^{2,2}(\Omega)$, using (5.1.2) and integrating by parts we find that

$$\int_\Omega u\, S_2(D^2u)\, dx = 3\int_\Omega \sum_{1\leqslant i<j\leqslant N} \partial_i u \partial_j u \partial_{ij} u \, dx. \tag{5.2.6}$$

Then, if $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{2,2}(\Omega)$ is a Palais-Smale sequence for $\mathcal{E}$ at level c, denoting $\langle y_n, u_n\rangle := \langle \mathcal{E}'(u_n), u_n\rangle$, it follows

$$\begin{aligned} c + o(1) &= \mathcal{E}(u_n) - \frac{1}{3}\langle \mathcal{E}'(u_n), u_n\rangle + \frac{1}{3}\langle y_n, u_n\rangle \\ &\geqslant \left(\frac{1}{2} - \frac{1}{3}\right)\int_\Omega |\Delta u_n|^2\,dx - \frac{1}{3}\|y_n\|_{W^{-2,2}(\Omega)}\left(\int_\Omega |\Delta u_n|^2\,dx\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{2} - \frac{1}{3}\right)\|\Delta u_n\|^2_{L^2(\Omega)} - \frac{1}{3}\|y_n\|_{W^{-2,2}(\Omega)}\|\Delta u_n\|_{L^2(\Omega)}. \end{aligned}$$

Then we easily conclude the existence of $C > 0$, independent of n, such that

$$\|\Delta u_n\|_{L^2(\Omega)} \leqslant C.$$

Step 4: By Lemma 5.2.1, there exists u such that

$$u_n \to u \ \text{ in } W_0^{2,2}(\Omega),$$

and therefore

(i) $\mathcal{E}(u) = \lim\limits_{n\to\infty} \mathcal{E}(u_n) = c.$

(ii) $\mathcal{E}'(u) = 0$, i.e., u satisfies

$$\Delta^2 u = S_2(D^2 u), \ u \in W_0^{2,2}(\Omega).$$

Hence, u is a solution to the problem (5.2.1). □

5.2.1 Influence of a reaction term in the multiplicity of solutions.

Consider now the problem

$$\text{(5.2.7)} \qquad \begin{cases} \Delta^2 u = S_2(D^2 u) + \mu |u|^{p-1} u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where the hypotheses assumed on (5.2.1) hold, and suppose $\mu > 0$ and $0 < p < \infty$. The idea now is to analyze how the reaction term affects the solvability of the problem. We will see that this influence will actually depend on the power p, obtaining different results for the three cases $p < 1$, $p = 1$ and $p > 1$.

First of all, we recall that this problem still has a variational formulation and, analogously to the previous case, it can be checked that the critical points of the functional

$$\mathcal{E}_\mu : W_0^{2,2}(\Omega) \to \mathbb{R},$$

defined by

$$\mathcal{E}_\mu(u) := \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \int_\Omega \sum_{1\leqslant i<j\leqslant N} \partial_{ij}u\partial_i u\partial_j u\,dx - \frac{\mu}{p+1}\int_\Omega |u|^{p+1}\,dx$$

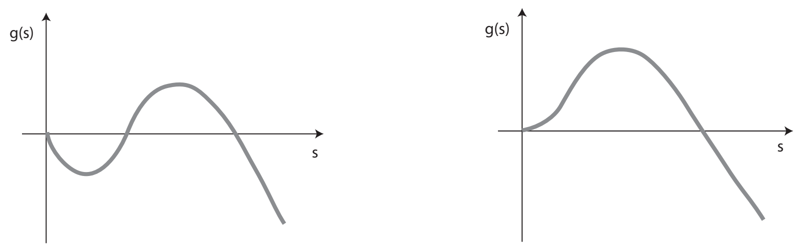
are indeed energy solutions of problem (5.2.7).

Since for $N=3$ the critical exponent in the Rellich-Kondrachov Theorem is $+\infty$, it can be easily proved that $\mathcal{E}_\mu$ satisfies the Palais-Smale condition for the whole range of p in a very similar way to the proof of Lemma 5.2.1. Concerning the geometry, in this case the functional satisfies

$$\begin{aligned}\mathcal{E}_\mu(u) &\geqslant \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - C\left(\int_\Omega |\Delta u|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^4\,dx\right)^{\frac{1}{2}} - \frac{\mu}{p+1}\int_\Omega |u|^{p+1}\,dx\\ &\geqslant \frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} - c_1\|\Delta u\|^3_{L^2(\Omega)} - \mu c_2\|\Delta u\|^{p+1}_{L^2(\Omega)} = g\left(\|\Delta u\|_{L^2(\Omega)}\right),\end{aligned}$$

where

$$g(s) = \frac{1}{2}s^2 - c_1 s^3 - \mu c_2 s^{p+1}, \quad s>0. \tag{5.2.8}$$



(a) $p<1$, μ small (b) $p\geqslant 1$

Clearly, the geometry of the functional depends on p, and this fact motivates the different results on existence and multiplicity that we will obtain on (5.2.7).

Theorem 5.2.4. *Let* $0<p<\infty$.

- *(i) If* $p<1$ *there exists a* $\mu_0>0$ *such that if* $0<\mu<\mu_0$, *problem* (5.2.7) *has at least two nontrivial solutions.*
- *(ii) If* $p>1$ *problem* (5.2.7) *has at least one nontrivial solution for every* $\mu\geqslant 0$.
- *(iii) If* $p=1$ *problem* (5.2.7) *has at least one nontrivial solution whenever* $0<\mu<\mu_1$, *where* μ_1 *denotes the first eigenvalue of* Δ^2 *in* Ω *with Dirichlet boundary conditions.*

Proof. We handle every case separately.

(i) $0<p<1$: As in the proof of Theorem 5.2.3, we proceed in several steps:

Step 1: It is easy to check that:

(a) There exists a function $\phi \in W_0^{2,2}(\Omega)$ such that

$$\int_\Omega |\phi|^{p+1}\, dx > 0.$$

(b) There exists a function $\psi \in W_0^{2,2}(\Omega)$ such that

$$\int_\Omega \sum_{1\leqslant i<j\leqslant N} \partial_{ij}\psi \partial_i \psi \partial_j \psi dx > 0.$$

Consequently,

$$\mathcal{E}_\mu(t\phi) < 0 \text{ for } t \text{ small enough and } \mathcal{E}_\mu(s\psi) < 0 \text{ for } s \text{ large enough.}$$

Step 2: It is easy to check that for $0 < \mu < \mu_0$ small enough the function g, defined in (5.2.8), has a local negative minimum and a local positive maximum. Hence, we will search for a local minimum and a *mountain pass* critical point of the functional.

Step 3: There exists μ_0 such that if $0 < \mu < \mu_0$, then $\mathcal{E}_\mu$ has a local minimum u_0, with $\mathcal{E}_\mu(u_0) < 0$.
We follow here the ideas of [112]. Take μ_0 such that g attains its positive maximum at $r_{max} > 0$, and denote by r_0 the lower positive zero of g. Fix r_1 such that $r_0 < r_1 < r_{max}$, with $g(r_1) > 0$. Define now a nonincreasing cutoff function $\tau \in C^\infty$, verifying

$$\tau : \mathbb{R}_+ \to [0,1],$$

$$\begin{cases} \tau(s) = 1, & \text{if } s \leqslant r_0 \\ \tau(s) = 0, & \text{if } s \geqslant r_1. \end{cases}$$

Consider now the truncated functional

$$\mathcal{E}_{\mu,\tau}(u) := \frac{1}{2}\int_\Omega |\Delta u|^2\, dx - \tau\left(\|\Delta u\|_{L^2(\Omega)}\right)\int_\Omega \sum_{1\leqslant i<j\leqslant N} \partial_{ij} u \partial_i u \partial_j u\, dx - \mu \int_\Omega |u|^{p+1}\, dx.$$

This functional is coercive in $W_0^{2,2}(\Omega)$, and thus it achieves a local minimum u_0 with negative energy. Consequently, u_0 is also a local minimum of $\mathcal{E}_\mu$, and $\mathcal{E}_\mu(u_0) < 0$.

Step 4: If $\mu < \mu_0$, $\mathcal{E}_\mu$ satisfies the Mountain Pass geometry.
Recalling together Step 1 and Step 2, the Mountain Pass geometry follows in a straightforward way.

Consider now the local minimum u_0 obtained in Step 3, that satisfies $\mathcal{E}_\mu(u_0) < 0$. Take also $v \in W_0^{2,2}(\Omega)$, with $\|\Delta v\|_{L^2(\Omega)} > r_{max}$, such that $\mathcal{E}_\mu(v) < \mathcal{E}_\mu(u_0)$.

We define

$$\Gamma := \{\gamma \in \mathcal{C}\left([0,1], W_0^{2,2}(\Omega)\right) \mid \gamma(0) = u_0,\, \gamma(1) = v\},$$

and the mini-max value

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{E}_\mu[\gamma(t)].$$

Applying the Ekeland variational principle (see [87]), we conclude that there exists a Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{2,2}(\Omega)$ at level c.

Step 5: If $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{2,2}(\Omega)$ is a Palais-Smale sequence for $\mathcal{E}_\mu$ at the level c, then there exists $C > 0$, independent of n, such that $\|\Delta u_n\|_{L^2(\Omega)} < C$. This can be proved reproducing Step 3 of the proof of Theorem 5.2.3.

Step 6: Since $\mathcal{E}_\mu$ satisfies the Palais-Smale condition, we can conclude that

$$u_n \to u \text{ in } W_0^{2,2}(\Omega),$$

being u a solution to the problem (5.2.7).

(ii) $1 < p < \infty$: Notice that the term corresponding to $S_2(D^2u)$ plays the role of a superlinear power in the functional. Hence, the term $\mu|u|^{p-1}u$, that in this case will be superlinear too, adds nothing new to the problem (5.2.1) concerning the solvability and multiplicity. The proof identically follows the outline of Theorem 5.2.3, so we skip it.

(iii) $p = 1$: In this case, the associated functional is

$$\mathcal{E}_\mu(u) := \frac{1}{2}\int_\Omega |\Delta u|^2 - \int_\Omega \sum_{1\leqslant i<j\leqslant N} \partial_{ij}u\partial_i u\partial_j u - \frac{\mu}{2}\int_\Omega |u|^2,$$

and the first thing we remark is that $\mathcal{E}_\mu$ satisfies

$$\mathcal{E}_\mu(u) \geqslant \left(\frac{1}{2} - \frac{\mu}{2}C_1\right)\|\Delta u\|^2_{L^2(\Omega)} - C_2\|\Delta u\|^3_{L^2(\Omega)},$$

where C_1 is the constant that appears in the Poincaré inequality

$$\left(\int_\Omega |u|^2\,dx\right)^{\frac{1}{2}} \leqslant C_1^{1/2}\left(\int_\Omega |\Delta u|^2\,dx\right)^{\frac{1}{2}}.$$

Hence, if $\mu < C_1^{-1}$, the geometry of the functional is essentially the same as in the superlinear case. However, if we remind that the first eigenvalue of the bilaplacian minimizes the Rayleigh quotient, that is,

$$\mu_1 = \inf_{u\in W_0^{2,2}(\Omega),u\neq 0}\left\{\frac{\int_\Omega |\Delta u|^2\,dx}{\int_\Omega |u|^2\,dx}\right\},$$

then this implies that necessarily $\mu < \mu_1$.

If this condition holds, in a very close way to the previous case, we can prove the existence of a nontrivial solution as a consequence of the Mountain Pass Lemma. □

5.3 The homogeneous problem with Navier conditions: bifurcation.

As a consequence of the Dirichlet boundary conditions, in the previous section we could work with the variational formulation associated to the problems, and hence the existence results were fully based on variational techniques. However, in the sequel we deal with problem (5.0.1) when we prescribe boundary conditions of Navier type. Thus, as we pointed out in Remark 5.1.7, we do not know whether this problem corresponds to the Euler-Lagrange equation associated to some energy functional, so we refuse to use variational methods here.

Furthermore, if we consider the problem

$$\begin{cases} \Delta^2 u = S_2(D^2 u), & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3.1}$$

we do not even know whether a nontrivial solution exists. Nonetheless, if we add a reaction term, i.e., if we study the problem

$$\begin{cases} \Delta^2 u = S_2(D^2 u) + \lambda u, & \Omega, \\ \quad u = 0, & \partial\Omega, \\ \Delta u = 0, & \partial\Omega, \end{cases} \tag{5.3.2}$$

we can use the bifurcation theory to obtain information about its solvability (see for example [23]). More precisely, we would like to apply the Rabinowitz global bifurcation theorem over this problem. Before stating this result, let us detail some necessary notation.

Let X be a Banach space, $A \in L(X)$ and $T \in C^1(X, X)$, and define

$$\mathcal{I}_\lambda(u) := u - \lambda A u - T u, \quad u \in X, \tag{5.3.3}$$

and

$$\Sigma := \{(\lambda, u) \in \mathbb{R} \times X,\, u \neq 0 : \mathcal{I}_\lambda(u) = 0\}.$$

We introduce first the notion of bifurcation point.

Definition 5.3.1. A bifurcation point for

$$\mathcal{I}_\lambda(u) = 0,$$

is a number $\lambda^* \in \mathbb{R}$ such that $(\lambda^*, 0)$ belongs to $\overline{\Sigma}$, that is, λ^* is a bifurcation point if there exist sequences $\lambda_n \in \mathbb{R}$, $u_n \in X \setminus \{0\}$ such that

(i) $I_{\lambda_n}(u_n) = 0$,

(ii) $(\lambda_n, u_n) \to (\lambda^*, 0)$.

Indeed, the Kranoselski bifurcation theorem (see [127]) ensures the existence of bifurcation points whenever the operators A and T satisfy certain conditions. More precisely,

Theorem 5.3.2. *(Krasnoselski, 1964)*
Let $A \in L(X)$ be compact and let $T \in C^1(X, X)$ be compact and such that $T(0) = 0$ and $T'(0) = 0$. Then, every characteristic value λ^* of A with odd multiplicity is a bifurcation point for

$$\mathcal{I}_\lambda(u) = 0.$$

And finally, once we know that $(\lambda^*, 0)$ is a bifurcation point, the Rabinowitz global bifurcation theorem (see [149]) provides information about the behavior of the branch arising from it.

Theorem 5.3.3. *(Rabinowitz, 1970)*
Let $A \in L(X)$ be compact and let $T \in C^1(X, X)$ be compact and such that $T(0) = 0$ and $T'(0) = 0$. Suppose that λ^* is a characteristic value of A with odd multiplicity. Let $\mathcal{C}$ be the connected component of $\overline{\Sigma}$ containing $(\lambda^*, 0)$. Then either

(i) $\mathcal{C}$ is unbounded in $\mathbb{R} \times X$, or

(ii) there exists $\hat{\lambda} \in \rho(A) \setminus \{\lambda^*\}$ such that $(\hat{\lambda}, 0) \in \mathcal{C}$.

In order to apply these theorems to (5.3.2), we previously need to study the eigenvalue problem associated to Δ^2,

$$(EP)_N \begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

As we explained in the Introduction, one particularity of these boundary conditions is that we can reformulate this problem as the following two Dirichlet problems for the laplacian,

$$\begin{cases} -\Delta u = v, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta v = \lambda u, & \text{in } \Omega, \\ \quad v = 0, & \text{on } \partial\Omega, \end{cases}$$

so that we can apply all the well known machinery associated to this framework. Consequently,

Proposition 5.3.4. *Let λ_1 be the first eigenvalue of the problem $(EP)_N$, that is,*

$$\lambda_1 := \inf_{u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),\, u \neq 0} \left\{ \frac{\int_\Omega |\Delta u|^2\, dx}{\int_\Omega u^2\, dx} \right\}.$$

Then, λ_1 is simple, i.e., the set of solutions to $(EP)_N$ with $\lambda = \lambda_1$ form a one dimensional subspace of $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

Proof. Consider the problem

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3.4}$$

It is well known that the first eigenvalue μ_1 of this problem is simple, and its associated eigenfunction, ϕ_1, positive. Moreover, the eigenfunctions ϕ_k of (5.3.4), with corresponding eigenvalues μ_k, set an orthonormal basis in $L^2(\Omega)$.

Since Ω is smooth, these eigenfunctions ϕ_k are actually $\mathcal{C}^\infty(\overline{\Omega})$, and then (5.3.4) holds pointwise, and it is easy to check that $\lambda_k = \mu_k^2$ are eigenvalues of $(EP)_N$, and ϕ_k the associated eigenfunctions. Indeed,

$$\Delta^2\phi_k = -\Delta(-\Delta\phi) = -\Delta\mu_k\phi_k = \mu_k^2\phi_k,$$

and $\Delta\phi_k = 0$ on $\partial\Omega$ by the Trace theorem. Finally, let us suppose that there exists $\tilde{\phi} \neq \phi_1$ such that

$$\begin{cases} \Delta^2\tilde{\phi} = \lambda_1\tilde{\phi}, & \text{in } \Omega, \\ \tilde{\phi} = \Delta\tilde{\phi} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, since $\{\phi_k\}_{k\in\mathbb{N}}$ form a basis of $L^2(\Omega)$, there exists a sequence $\{a_k\}_{k\in\mathbb{N}}$ of parameters such that

$$\tilde{\phi} = \sum_k a_k\phi_k.$$

Thus,

$$\lambda_1\sum_k a_k\phi_k = \lambda_1\tilde{\phi} = \Delta^2\tilde{\phi} = \sum_k a_k\Delta^2\phi_k = \sum_k a_k\lambda_k\phi_k,$$

and therefore

$$\sum_k a_k(\lambda_1 - \lambda_k)\phi_k = 0.$$

Using the orthogonality of $\{\phi_k\}_{k\in\mathbb{N}}$ we conclude that necessarily $a_k = 0$ for every $k \neq 1$, that is, $\tilde{\phi} = a_1\phi_1$, and we conclude that λ_1 is simple.

By a very close argument, one can prove that there does not exist an eigenvalue of $(EP)_N$ different from $\lambda_k = \mu_k^2$, and thus $\lambda_1 = \mu_1^2$ is its first eigenvalue. Actually, this fact is a consequence of the spectral theory for $-\Delta$. $\square$

As a consecuence of this property for the first eigenvalue and the two previous results on bifurcation theory, we can already give some information about the solvability of problem (5.3.2).

Theorem 5.3.5. *Let λ_1 be the first eigenvalue of Δ^2 in Ω with Navier boundary conditions. Then, there exists an unbounded branch of pairs (λ, u), bifurcating from $(\lambda_1, 0)$, where every $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a solution to (5.3.2) with the corresponding λ.*

Proof. Let us define $X := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, and $A, T : X \to X$ as

$$Au := \Delta^{-2}u, \text{ and } Tu := A(S_2(D^2u)).$$

Hence,

$$\Delta^2 u = S_2(D^2u) + \lambda u$$

is equivalent to

$$\mathcal{I}_\lambda(u) = 0,$$

with I_λ defined in (5.3.3). To apply the Krasnoselski and Rabinowitz Theorem we first need to check that A and T are compact from $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ to $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$. Consider the problem

$$\begin{cases} \Delta^2 u = S_2(D^2\varphi), & \text{in } \Omega,\\ u=\Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\varphi\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$. Since we are dealing with Navier conditions, we can split this problem into

$$\begin{cases} -\Delta u = v, & \text{in } \Omega,\\ \quad u=0, & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta v = S_2(D^2\varphi), & \text{in } \Omega,\\ \quad v=0, & \text{on } \partial\Omega. \end{cases}$$

By Lemma 5.1.4, $S_2(D^2\varphi)\in L^1(\Omega)$ and hence, considering the second problem, we deduce $v\in W_0^{1,q}(\Omega)$ for every $q<\frac{3}{2}$. Thus $u\in W^{3,q}(\Omega)\cap W_0^{1,2}(\Omega)$.

As a consequence, $\Delta u\in W^{1,q}(\Omega)$, and by Rellich-Kondrachov theorem, this space is compactly embedded in $L^p(\Omega)$, for $p<\frac{3q}{3-q}$. In particular, taking q close enough to $3/2$, we can choose $p=2$, that is, $W^{1,q}(\Omega)\subset\subset L^2(\Omega)$.

Therefore, if we have a sequence $\{u_k\}_{k\in\mathbb{N}}$ uniformly bounded in $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$, there exists a limit $u\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ such that

$$\Delta^2 u_k \to \Delta u \text{ in } L^2(\Omega),$$

i.e., T is a compact operator. Likewise, reasoning in the same way with the problem

$$\begin{cases} \Delta^2 u = \lambda\varphi, & \text{in } \Omega,\\ u=\Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

one can prove that A is also a compact operator.

On the other hand, we need to prove that $T(0)=0$ and $T'(0)=0$. Notice that $T(0)=A(0)$, so we get that $T(0)=0$. For the derivative, it can be checked that

$$\left.\frac{dT(u+t\varphi)}{dt}\right|_{t=0} = A\left(\sum_{1\leqslant i<j\leqslant N} \partial_{ii}u\partial_{jj}\varphi+\partial_{ii}\varphi\partial_{jj}u-2\partial_{ij}u\partial_{ij}\varphi\right),$$

and therefore we conclude $T'(0)=0$.

Finally, by Proposition 5.3.4, we know that the first eigenvalue λ_1 associated to A in Ω is simple, that is, it has multiplicity one. Thus, by Theorem 5.3.2 we know that λ_1 is a bifurcation point, and we can apply Theorem 5.3.3 to conclude that the connected component of $\overline{\Sigma}$ that contains $(\lambda_1,0)$ is unbounded, what follows from [149, Theorem 2.12]. $\square$

Remark 5.3.6. *Notice that in dimension $N=4$ we do not have compactness. For $N>4$ the situation is even worse, because the inverse operator $\Delta^{-2}F$ has a range in general greater than $W^{2,2}(\Omega)$.*

Remark 5.3.7. *We cannot use the Rabinowitz bifurcation theorem in the case of Dirichlet boundary conditions because the first eigenvalue is not simple in general. In fact, examples of domains with associated eigenvalues of even multiplicity can be found in the literature, see for instance [71]. However, J. H. Ortega and E. Zuazua proved in [146] that* λ_1 *is generically simple in the sense of* $W^{5,\infty}(\Omega)$ *deformations of* Ω.

5.4 The nonhomogeneous problem with Navier conditions: a fixed point argument.

Despite the fact that we are not able to solve problem (5.3.1), in the previous section we studied, by means of the bifurcation theory, the behavior of such problem when a reaction term, linear in u, appears. To complement this information, the aim of this section is to prove existence of solution of the nonhomogeneous problem associated to (5.3.1), that is,

$$\begin{cases} \Delta^2 u = S_2(D^2 u) + \mu f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4.1}$$

where μ is a positive parameter, and $f \in L^1(\Omega)$. More precisely, the main result of this section is the following.

Theorem 5.4.1. *Let* $f \in L^1(\Omega)$. *Then, there exists* $\mu_0 > 0$ *such that for every* μ *satisfying* $0 < \mu < \mu_0$, *there exists* $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ *solution to problem* (5.4.1).

The idea to prove this result will be to apply the Banach Fixed Point Theorem over the space $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. To do so, we will need use the following auxiliar result, that is just a reformulation of the two dimensional result proved in [91, Lemma 4.1].

Lemma 5.4.2. *For any functions* $v_1,\ v_2 \in W^{1,2}(\Omega)$ *and* $v_3 \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ *the following equality is fulfilled*

$$\int det(\nabla_{ij} v_1, \nabla_{ij} v_2) v_3 \, dx = \int v_1 \nabla_{ij} v_2 \cdot \nabla_{ij}^{\perp} v_3 \, dx$$

where $\nabla_{i,j} := (\partial_i, \partial_j)$ *and* $\nabla_{ij}^{\perp} := (\partial_j, -\partial_i)$.

With this technical lemma we can already prove Theorem 5.4.1.

Proof. Let us consider first the linear problems

$$\begin{cases} \Delta^2 u_1 = S_2(D^2 \varphi_1) + \mu f(x) & \text{in } \Omega, \\ u_1 = 0, \ \ \Delta u_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4.2}$$

and

$$\begin{cases} \Delta^2 u_2 = S_2(D^2 \varphi_2) + \mu f(x) & \text{in } \Omega, \\ u_2 = 0, \ \ \Delta u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4.3}$$

where $\varphi_1, \varphi_2 \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$. By Lemma 5.1.4, the right hand sides of these problems belong to $L^1(\Omega)$, and hence there exist solutions $u_1,\, u_2 \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ to (5.4.2) and (5.4.3) respectively. Notice that, since $N = 3$, with the only information $S_2(D^2\varphi_i) \in L^1(\Omega)$ is enough to start. Indeed, in such a case $u_i \in W^{3,q}(\Omega) \cap W_0^{1,2}(\Omega)$, for every $q < \frac{3}{2}$, and then, by the Sobolev embedding, $\Delta u_i \in L^2(\Omega)$.

Substracting both equations we obtain

$$\begin{cases} \Delta^2(u_1 - u_2) = S_2(D^2\varphi_1) - S_2(D^2\varphi_2) & \text{in } \Omega, \\ \quad u_1 - u_2 = 0, \quad \Delta(u_1 - u_2) = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.4.4}$$

Again, following [91], we know that

$$\begin{aligned} \det(D^2_{ij}(\varphi_1)) - \det(D^2_{ij}(\varphi_2)) = {} & \det\{\nabla_{ij}(\partial_i\varphi_1),\, \nabla_{ij}[\partial_j\varphi_1 - \partial_j\varphi_2]\} \\ & + \det\{\nabla_{ij}[\partial_i\varphi_1 - \partial_i\varphi_2],\, \nabla_{ij}(\partial_j\varphi_2)\}, \end{aligned}$$

and therefore, by Lemma 5.4.2, if $w \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$,

$$\begin{aligned} &\langle S_2(D^2\varphi_1) - S_2(D^2\varphi_2),\, w\rangle \\ &\qquad = \sum_{1\leqslant i<j\leqslant N} \langle \det(D^2_{ij}(\varphi_1)) - \det(D^2_{ij}(\varphi_2)),\, w\rangle \\ &\qquad = \sum_{1\leqslant i<j\leqslant N} \langle \det\{\nabla_{ij}(\partial_i\varphi_1),\, \nabla_{ij}[\partial_j\varphi_1 - \partial_j\varphi_2]\} \\ &\qquad\qquad\quad + \det\{\nabla_{ij}[\partial_i\varphi_1 - \partial_i\varphi_2],\, \nabla_{ij}(\partial_j\varphi_2)\},\, w\rangle \\ &\qquad = \sum_{1\leqslant i<j\leqslant N} \int [\partial_i\varphi_1 \nabla_{ij}(\partial_j\varphi_1 - \partial_j\varphi_2) - \partial_j\varphi_2 \nabla_{ij}(\partial_i\varphi_1 - \partial_i\varphi_2)] \cdot \nabla^{\perp}_{ij} w\, dx. \end{aligned}$$

Consequently,

$$\begin{aligned} &|\langle S_2(D^2\varphi_1) - S_2(D^2\varphi_2),\, w\rangle| \\ &\qquad \leqslant \sum_{1\leqslant i<j\leqslant N} \int [|\nabla\varphi_1||D^2_{ij}(\varphi_1 - \varphi_2)| + |\nabla\varphi_2||D^2_{ij}(\varphi_1 - \varphi_2)|]|\nabla_{ij} w|\, dx \\ &\qquad \leqslant \sum_{1\leqslant i<j\leqslant N} \int (|\nabla\varphi_1| + |\nabla\varphi_2|)|D^2(\varphi_1 - \varphi_2)||\nabla w|\, dx \\ &\qquad \leqslant C(N)\left(\|\nabla\varphi_1\|_{L^4(\Omega)} + \|\nabla\varphi_2\|_{L^4(\Omega)}\right)\|D^2(\varphi_1 - \varphi_2)\|_{L^2(\Omega)}\|\nabla w\|_{L^4(\Omega)}, \end{aligned}$$

and finally, using the Sobolev embedding for $W_0^{1,2}(\Omega)$,

$$\begin{aligned} &|\langle S_2(D^2\varphi_1) - S_2(D^2\varphi_2),\, w\rangle| \\ &\qquad \leqslant C(N)\left(\|\Delta\varphi_1\|_{L^2(\Omega)} + \|\Delta\varphi_2\|_{L^2(\Omega)}\right)\|\Delta(\varphi_1 - \varphi_2)\|_{L^2(\Omega)}\|\Delta w\|_{L^2(\Omega)}. \end{aligned}$$

Thus, testing in (5.4.4) with $(u_1 - u_2)$ and noticing that $(u_1 - u_2) \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, we obtain

$$\begin{aligned}\int |\Delta(u_1-u_2)|^2\,dx &\leqslant |\langle S_2(D^2\varphi_1)-S_2(D^2\varphi_2),\, u_1-u_2\rangle| \\ &\leqslant C(N)\left(\|\Delta\varphi_1\|_{L^2(\Omega)}+\|\Delta\varphi_2\|_{L^2(\Omega)}\right)\|\Delta(\varphi_1-\varphi_2)\|_{L^2(\Omega)}\|\Delta(u_1-u_2)\|_{L^2(\Omega)},\end{aligned}$$

that is,

$$\begin{aligned}&\|\Delta(u_1-u_2)\|_{L^2(\Omega)} \\ &\qquad\leqslant C\left(\|\Delta\varphi_1\|_{L^2(\Omega)}+\|\Delta\varphi_2\|_{L^2(\Omega)}\right)\|\Delta(\varphi_1-\varphi_2)\|_{L^2(\Omega)}.\end{aligned}\tag{5.4.5}$$

Let now v be the solution to the problem

$$\begin{cases}\Delta^2 v=\mu f(x) & \text{in } \Omega,\\ \quad v=0,\ \ \Delta v=0, & \text{on } \partial\Omega.\end{cases}\tag{5.4.6}$$

By the Sobolev embedding, there holds

$$\|\Delta v\|_{L^2(\Omega)}\leqslant \mu\|f\|_{L^1(\Omega)}.$$

Let $\rho>0$ to be chosen later, and suppose that

$$\varphi_1,\,\varphi_2\in B_\rho(v):=\{\varphi\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega):\ \|\Delta(v-\varphi)\|_{L^2(\Omega)}\leqslant\rho\}.$$

Hence, equation (5.4.5) becomes

$$\begin{aligned}\|\Delta(u_1-u_2)\|_{L^2(\Omega)}&\leqslant C(\rho+\|\Delta v\|_{L^2(\Omega)})\|\Delta(\varphi_1-\varphi_2)\|_{L^2(\Omega)}\\ &\leqslant\frac12\|\Delta(\varphi_1-\varphi_2)\|_{L^2(\Omega)},\end{aligned}\tag{5.4.7}$$

for μ and ρ small enough. Moreover, for both $i=1$ and $i=2$, by (5.4.2), (5.4.3) and (5.4.6),

$$\begin{aligned}\|\Delta(u_i-v)\|^2_{L^2(\Omega)}=\int\Delta(u_i-v)\Delta(u_i-v)\,dx&=\int S_2(D^2\varphi_i)(u_i-v)\,dx\\ &\leqslant\|u_i-v\|_{L^\infty(\Omega)}\|S_2(D^2\varphi_i)\|_{L^1(\Omega)},\end{aligned}\tag{5.4.8}$$

and hence, from this and the Sobolev embedding, we conclude

$$\|\Delta(u_i-v)\|_{L^2(\Omega)}\ \leqslant\ \tilde C\|\Delta\varphi_i\|^2_{L^2(\Omega)}.$$

On the other hand,

$$\begin{aligned}\|\Delta\varphi_i\|^2_{L^2(\Omega)}&\leqslant\ 2\left(\|\Delta(\varphi_i-v)\|^2_{L^2(\Omega)}+\|\Delta v\|^2_{L^2(\Omega)}\right)\\ &\leqslant\ 2\left(\rho^2+\mu^2\|f\|^2_{L^1(\Omega)}\right)\leqslant\frac{\rho}{\tilde C}\end{aligned}$$

for ρ and μ small enough. Thus,

$$\|\Delta(u_i - v)\|_{L^2(\Omega)} \leqslant \rho, \tag{5.4.9}$$

and hence $u_i \in B_\rho(v)$. Therefore, if we define the operator

$$\begin{array}{rcl} T : B_\rho(v) & \to & B_\rho(v) \\ \varphi & \to & T(\varphi) = u, \end{array}$$

where u is the solution to the problem

$$\begin{cases} \Delta^2 u = S_2(D^2\varphi) + \mu f(x) & \text{in } \Omega, \\ \quad u = 0, \ \ \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

inequality (5.4.7) assures that T is contractive, and hence, by Banach Fixed Point Theorem, there exists a unique fixed point $u = T(u)$, that is a solution to problem (5.4.1). $\square$

Remark 5.4.3. *This argument can be exactly reproduced in the case of Dirichlet boundary conditions for $N = 3$. Moreover, thanks to the Sobolev embedding, the same result can be obtain if we add a reaction term $|u|^{p-1}u$, with $p > 1$, in problem* (5.4.1).

Remark 5.4.4. *This proof cannot be extended to the case $N = 4$, because it strongly depends on the embedding $W^{2,2}(\Omega) \subset L^\infty(\Omega)$. Indeed, for dimension $N = 3$, the embedding $W^{2,2}(\Omega) \subset L^p(\Omega)$ is obtained for all $p \leqslant +\infty$. However, the inclusion in the extremal case $p = +\infty$ is not reached for $N = 4$.*

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